THE 2006 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

February 7, 2007
Exam Time – 2 Hours

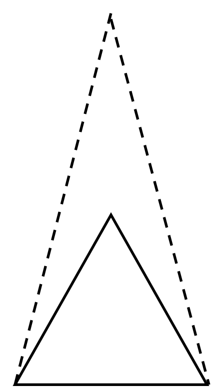
NO CALCULATORS

1. Prove that $x^2 - y^2 = A^3$ always has integer solutions (x, y) whenever A is a positive integer.

2. In non-cyclic quadrilateral ABCD, points E, F, G, H are the circumcenters of triangles ABC, BCD, CDA, and DAB, respectively. The diagonals of quadrilateral EFGH meet at point P. Prove that $\overline{AP} \cong \overline{CP}$ and $\overline{BP} \cong \overline{DP}$.

3. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ be a polynomial with integer coefficients. Suppose there exist distinct integers a, b, c, d such that $P(a) = P(b) = P(c) = P(d) = 4$. Prove that there exists <u>no</u> integer m such that $P(m) = 7$.

4. The lengths of two sides of an equilateral triangle are doubled, creating an isosceles triangle, as shown in the diagram at the right. These two longer sides are doubled again creating a third isosceles triangle (all three triangles having the same base). This process is continued indefinitely. If the measure of the vertex angle of each triangle is represented by $A_1, A_2, A_3, \ldots$, determine, with proof, the value of $(1-\cos A_1) + (1-\cos A_2) + (1-\cos A_3) + \ldots$.



5. Let S be a sequence of consecutive positive integers whose terms have a sum of 2007. Find all such sequences S, and prove that you have found them all. (Do <u>not</u> include the trivial sequence consisting of 2007 alone.)

SOLUTIONS – KSU MATHEMATICS COMPETITION – PART II 2006-07

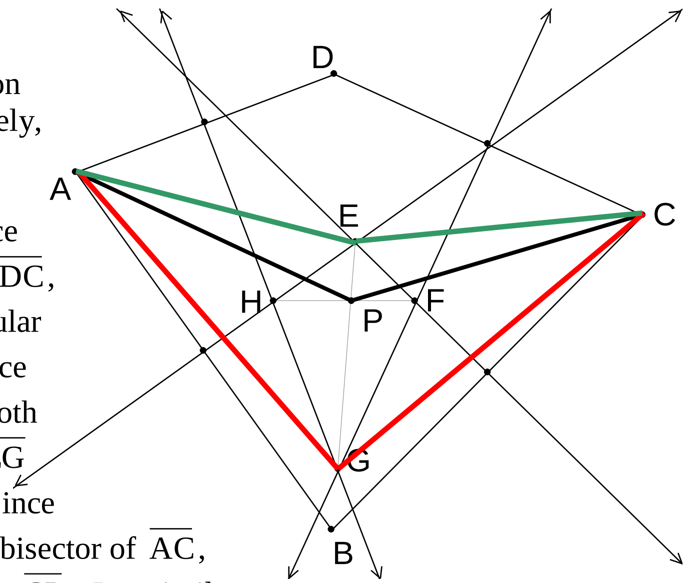
1. Since $x^2 - y^2 = (x+y)(x-y) = A^3$, let's set $x + y = A^2$ and $x - y = A$

   Adding both of these equations, we obtain $2x = A^2 + A$ and $x = \frac{A(A+1)}{2}$.

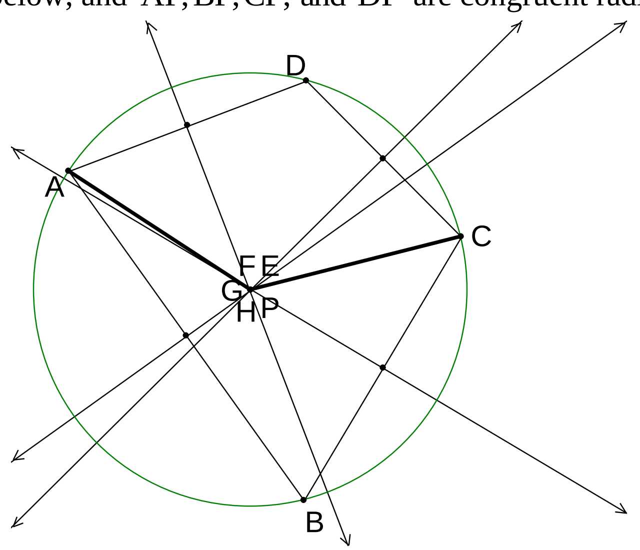
   Subtracting the two equations, we obtain $2y = A^2 - A$ and $y = \frac{A(A-1)}{2}$.

   For any positive integer A, both A(A-1) and A(A+1) are products of 2 consecutive integers and are, therefore, both even. Hence, $x = \frac{A(A+1)}{2}$ and $y = \frac{A(A-1)}{2}$ are a pair of integer solutions to the equation.

2. The circumcenter of a triangle is the intersection of the perpendicular bisectors of its sides, namely, points E,F, G, and H. Any point on the perpendicular bisector of a segment is equidistant from the segment's endpoints. Since G is on the perpendicular bisector of $\overline{AD}$ and $\overline{DC}$, $\overline{AG} \cong \overline{DG} \cong \overline{CG}$. Since E is on the perpendicular bisector of $\overline{AB}$ and $\overline{BC}$, $\overline{AE} \cong \overline{EB} \cong \overline{EC}$. Since $\overline{AG} \cong \overline{CG}$ and $\overline{AE} \cong \overline{EC}$ points E and G are both equidistant from points A and C. Therefore, $\overline{EG}$ lies along the perpendicular bisector of $\overline{AC}$. Since point P lies on $\overline{EG}$, it is also on perpendicular bisector of $\overline{AC}$, making P equidistant from A and C. Thus $\overline{AP} \cong \overline{CP}$**.** In a similar manner**,** we can prove $\overline{BP} \cong \overline{DP}$.



Note: If quadrilateral ABCD is cyclic the result still holds. In any circle, the perpendicular bisectors of the chords meet at the circle's center. Thus, if ABCD is a cyclic quadrilateral, points E, F, G, H, and P coincide with the circles' center, so that the quadrilateral degenerates to a point, as shown below, and $\overline{AP}, \overline{BP}, \overline{CP}$, and $\overline{DP}$ are congruent radii.

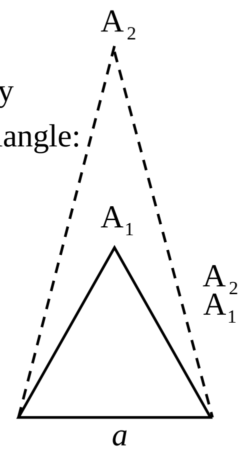


3. Since $P(a) = P(b) = P(c) = P(d) = 4$, the polynomial $f(x) = P(x) - 4$ has a, b, c, and d as zeros. Therefore, $f(x) = P(x) - 4 = (x - a)(x - b)(x - c)(x - d)g(x)$, where $g(x)$ is a polynomial with integer coefficients.

Suppose there exists an integer m such that $P(m) = 7$. Then

$$f(m) = P(m) - 4 = 7 - 4 = 3 = (m - a)(m - b)(m - c)(m - d)g(m).$$

Then the integers m – a, m – b, m – c, m – d, are distinct divisors of 3. Therefore, they must be equal to ±1 or ±3. However, only one of them can be ±3 or else their product could not be equal to 3. Hence, at least three of the numbers m – a, m – b, m – c, m – d must equal ±1. But then at least two of them are equal, which is a contradiction, since we are told they are distinct. Therefore, there exists no integer m such that $P(m) = 7$.



4. Let a represent the lengths of the sides of the equilateral triangle. Let the congruent sides of the isosceles triangle with vertex angle A_n be represented by b_n (noting that $a = b_1$). Then using the Law of cosines on the n^{th} isosceles triangle:

$$a^2 = (b_n)^2 + (b_n)^2 - 2(b_n)(b_n)\cos A_n = 2(b_n)^2 - 2(b_n)^2 \cos A_n$$

Noting that $b_n = (2^{n-1})a$, we have

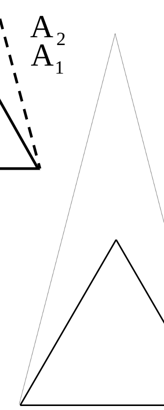
$$a^2 = 2\left[(2^{n-1})a\right]^2 - 2\left[2^{n-1}a\right]^2 \cos A_n = 2^{2n-1}a^2 - 2^{2n-1}a^2 \cos A_n$$

Therefore, $\cos A_n = \dfrac{2^{2n-1} - 1}{2^{2n-1}} = 1 - \dfrac{1}{2^{2n-1}}$ for all $n \geq 1$.

Thus, the desired sum becomes $\displaystyle\sum_{n=1}^{\infty}\left[1 - \left(1 - \frac{1}{2^{n-1}}\right)\right] = \sum_{n=1}^{\infty}\frac{1}{2^{2n-1}} = \frac{1}{2^1} + \frac{1}{2^3} + \frac{1}{2^5} + ...$

This is a geometric series with a first term a = $\dfrac{1}{2}$ and a ratio of r = $\dfrac{1}{4}$. Therefore,

the sum, S, of the series is $\dfrac{a}{1-r} = \dfrac{\frac{1}{2}}{1 - \frac{1}{4}} = \mathbf{\dfrac{2}{3}}$.



5. Let such a sequence be (m + 1), (m + 2), . . . , (m + k). The sum of the terms can be written as

$$km + \frac{k(k+1)}{2} = \frac{k(2m+k+1)}{2}.$$

Thus $\frac{k(2m+k+1)}{2} = 2007 = 3^2 \cdot 223$ and $k(2m + k + 1) = 18 \cdot 223$.

Since 223 is prime and is a factor of the product $k(2m + k + 1)$, then 223 divides either k or $2m + k + 1$. Since $1 + 2 + 3 + . . . + 63 = 2016 > 2007$, then $k < 63$. Therefore, 223 divides $2m + k + 1$ and k must be a factor of 18. Thus, k = 18, 9, 6, 3, 2, or 1.

If k = 18, then $2m + k + 1 = 223$ implies m = 102. This gives the sequence
103, 104, 105, . . ., 120.

If k = 9, then $2m + k + 1 = 2 \cdot 223 = 446$, and m = 218. This gives the sequence
219, 220, 221, . . ., 227.

If k = 6, then $2m + 2t + 1 = 3 \cdot 223$ implies m = 331. This gives the sequence
332, 333, 334, . . , 337.

If k = 3, then $2m + k + 1 = 6 \cdot 223 = 1338$, and m = 667. This gives the sequence
668, 669, 670.

If k = 2, then $2m + 2t +1 = 9 \cdot 223$ implies m = 1002. This gives the sequence
1003, 1004.

If k = 1, then $2m + k + 1 = 18 \cdot 223 = 4014$ implies m = 2006. This gives the sequence with one number, 2007, which we are asked not to include.

Thus, there are five sequences of consecutive positive integers whose terms sum to 2007:

103, 104, 105, . . . , 120
219, 220, 221, . . . , 227
332, 333, 334, . . . , 337
668, 669, 670
1003, 1004