THE 2008-2009 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

1. Let S be the set of all integers of the form $P^2 - 1$ where P is a prime number greater than 5. Let N be the largest integer that divides every member of S. Find, with proof, the value of N.

2. In triangle ABC, (cosB)(cosC) = cosA. Find, with proof, the numerical value of (tanB)(tanC).

3. Suppose that $n^2 + 1$ boys are lined up shoulder-to-shoulder from left to right in a straight line. Prove that it is always possible to select n + 1 boys to take one step forward so that, going from left to right, their heights are either consistently increasing or consistently decreasing.

4. The lengths of the sides of triangle ABC are in the ratio of 4:5:6. The bisector of the largest angle of the triangle is drawn. Prove that one of the two triangles formed also has sides whose lengths are in the ratio of 4:5:6.

5. All the factors of the polynomial $P(x) = x^3 + ax^2 + bx + b$ are linear with integer coefficients, and neither a nor b are zero or one. Find all possible pairs (a,b), and prove that you have found them all.

SOLUTIONS – KSU MATHEMATICS COMPETITION – PART II 2008–09

1. $P^2 - 1 = (P - 1)(P + 1)$. Since P is a prime and $P > 5$, P is odd. Therefore, both $P - 1$ and $P + 1$ must be consecutive even numbers. Therefore, one of them must also be a multiple of 4, which means $P^2 - 1$ is divisible by 8. Also, since $P - 1$, P and $P + 1$ are three consecutive integers, one of them must be a multiple of 3. Since P is prime, either $P - 1$ or $P + 1$ must be a multiple of 3. Therefore, $P^2 - 1$ is divisible by $8 \cdot 3 = 24$. If $P = 7$, $P^2 - 1 = 48$. If $P = 11$, $P^2 - 1 = 120$. Since 24 is the gcd of 48 and 120, **N = 24** is largest.

2. Since $A = 180 - (B + C)$, $(\cos B)(\cos C) = \cos A = \cos(180 - B - C)$.

   Using the formula for the cosine of the difference between two angles twice,

$$(\cos B)(\cos C) = \cos(180 - B)(\cos C) + \sin(180 - B)(\sin C)$$

$$= -(\cos B)(\cos C) + (\sin B)(\sin C)$$

   Therefore, $2(\cos B)(\cos C) = (\sin B)(\sin C)$ so that

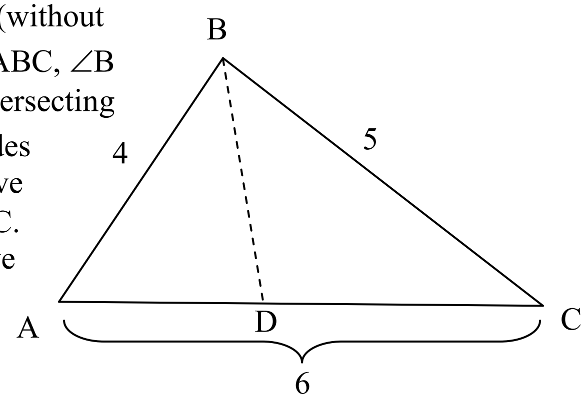
$$\frac{(\sin B)(\sin C)}{(\cos B)(\cos C)} = (\tan B)(\tan C) = \mathbf{2}.$$

3. Assume it is impossible to find n + 1 boys in ascending order of height from left to right. We will show that it is then possible to find n + 1 boys in descending order of height. Starting with the first boy, we form a "club" in this way: it consists of the first boy, the first boy taller than him, the first boy taller than <u>him</u>, etc. By the hypothesis, there are no more than n boys in this club. Next we create a second "club" by starting with the first boy not in the first club, and choosing the first boy who is not in the first club who is taller than the first boy in the second club, and continuing in this way. Again, there can be no more than n boys in the second club. Note that each boy in the second club is behind (to the right of) a boy in the first club who is taller than him (otherwise, he would have been in the first club). Now, from among the boys who are not in either of the first two clubs, we choose a third club starting with the first boy not in the first or second club. As before, there are no more than n boys in the third club, and each of these follows some member of the second club who is taller then him. We continue in this way, producing n clubs. Each of them has no more than n members, and each member of each club follows a taller boy in the previous club.

   There are at most n^2 members in these n clubs, so now let us start with any boy who is not in any of these clubs. He must follow a taller boy in the n^{th} club, who follows a still taller boy in the $(n-1)^{th}$ club, who follows a still taller boy in the $(n-2)^{th}$ club, etc. This gives us n + 1 boys, with the tallest first, the shortest last, and the others decreasing in order.

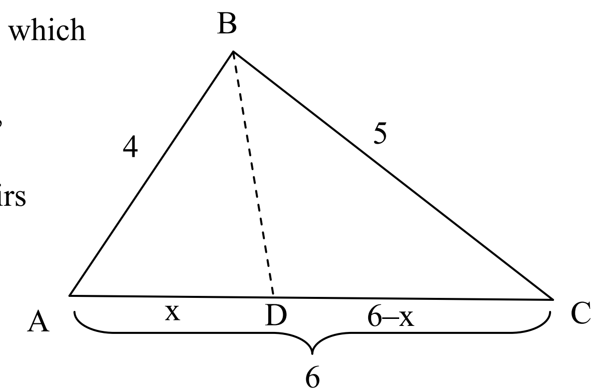
   If our original assumption had been that it is impossible to find n + 1 boys in descending order of height from left to right, a similar argument could be used to show that it is possible to find n + 1 boys in ascending order of height.

4. Let the sides of triangle ABC have lengths 4, 5, and 6 (without loss of generality). Since $\overline{AC}$ is the longest side of ΔABC, $\angle B$ is the largest angle. Draw the angle bisector of $\angle B$ intersecting $\overline{AC}$ at point D. Since the lengths of corresponding sides of similar triangles are proportional, we need only prove that one of the two triangles formed is similar to ΔABC. Since both triangles ABC and ADB are acute, we prove that these two triangles are similar.



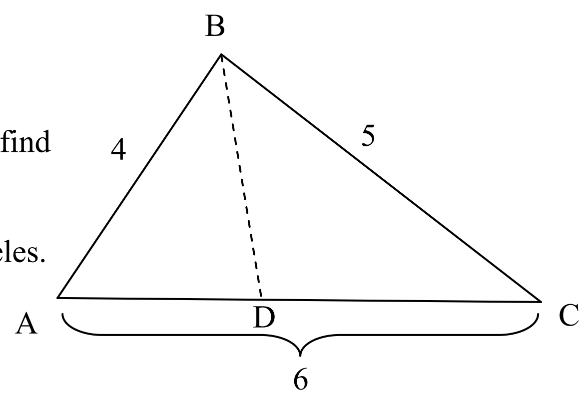
<u>Method 1</u>:

Using the angle bisector theorem, $\frac{4}{5} = \frac{x}{6-x}$ from which $x = \frac{8}{3}$. Since the ratio of AB to AD is $4:\frac{8}{3}$ or 6:4, $\frac{AB}{AD} = \frac{AC}{AB}$. Thus, ΔADB and ΔABC have two pairs of proportional sides and share the included angle A. Therefore, the two triangles are similar, proving ΔADB also has sides whose lengths are in the ratio of 4:5:6.



<u>Method 2</u>:

Since ΔADB and ΔABC share angle A, we need only find one additional pair of congruent angles. Angle ABD cannot be congruent to angle ABC. If it is congruent to angle C, then triangle BDC would have to be isosceles. This can only happen if the measure of angle ABC is twice that of angle C.



Using the Law of Cosines on ΔABC:

$$6^2 = 4^2 + 5^2 - 2(4)(5)\cos(\angle ABC) \;\Rightarrow\; \cos\angle ABC = \frac{1}{8}$$

$$4^2 = 6^2 + 5^2 - 2(6)(5)\cos C \;\Rightarrow\; \cos C = \frac{3}{4}$$

$\cos(2C) = 2\cos^2 C - 1 = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}$. Therefore, $m\angle ABC = 2(m\angle C)$.

Thus, triangle ABC is similar to triangle ADB, proving ΔADB also has sides whose lengths are in the ratio of 4:5:6.

5. Let $P(x) = x^3 + ax^2 + bx + b = (x+m)(x+n)(x+p) = x^3 + (m+n+p)x^2 + (mn + mp + np)x + mnp$, where m, n , and p are integers. We have

$$(1) \quad a = m + n + p \quad \text{and} \quad (2) \quad b = mn + mp + np = mnp.$$

Since $b \neq 0$, none of m, n, p can be 0. We can also show that none of m, n, p can be 1. Suppose, say, m = 1. By equation (2), $1n + 1p + np = 1np$. This reduces to $n = -p$. Therefore, by equation (1), $a = 1 - p + p = 1$, yet we are given $a \neq 1$.

Divide both sides of equation (2) by mnp and obtain (3) $\frac{1}{p} + \frac{1}{n} + \frac{1}{m} = 1$.

We now show that none of m, n, p can be negative. Suppose, for example, p is negative. Then $\frac{1}{n} + \frac{1}{m} = 1 - \frac{1}{p}$ gives an equation with left side less than 1 and right side more than 1.

We are now looking for solutions to equation (3) with m, n, p all at least 2.

Suppose that $m \leq n \leq p$. Trying m = 2 we get $\frac{1}{p} + \frac{1}{n} = \frac{1}{2}$ or $n = \frac{2p}{p-2}$. Noting that if $p > 6$, $(p - 2)$ can not be a factor of 2p, the only integer solutions (n,p) to this last equation are (3,6) or (4,4). Trying m = 3 (and remembering $m \leq n \leq p$) gives the solution $n = p = 3$.

With $m > 3$, $\frac{1}{p} + \frac{1}{n} + \frac{1}{m} < 1$, so there are no other solutions.

Using $a = m + n + p$ and $b = mnp$, we obtain $(a,b) =$ **(9,27), (10,32), (11,36)**.