THE 2009–2010 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box, or more than one box, is marked. Zero points are given for an incorrect answer. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. Let *l* be a line whose slope is equal to its y-intercept. There is one point in the x-y plane through which *l* must pass. What are the coordinates of this point?

(A) (1, 0) (B) (0, 1) (C) (1, 1) (D) (-1, 0) E (0, -1)

2. Let M be the product of the first 100 primes. How many zeros are at the end of M?

(A) 0 (B) 1 (C) 10 (D) 20 (E) 100

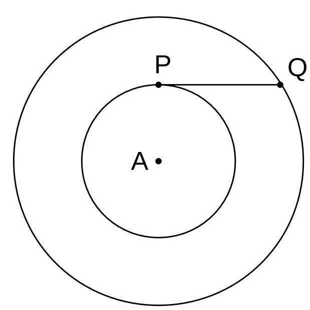
3. In Mr. Garner's Math class, he noted that when the attendance was exactly 84%, then there were 13 empty seats. If E represents the number of empty seats when all students attend, compute the largest possible value of E.

(A) 5 (B) 7 (C) 8 (D) 9 (E) 10

4. A three digit number has, from left to right, the digits h, t, and u, with h > u. When the number with the digits reversed is subtracted from the original number, the units digit of the difference is 4. Compute the difference.

(A) 324 (B) 414 (C) 594 (D) 684 (D) 774

5. Let A be the centers of two concentric circles, as shown. $\overline{PQ}$ is a line segment tangent to the smaller circle at point P and intersecting the larger circle at point Q. If PQ = 10, what is the area of the region lying between the two circles?



(A) 10π (B) 25π (C) 50π (D) 64π (E) 100π

6. How many different arrangements can be made using all the letters of the name **KENNESAW**?

(A) 40,320 (B) 20,160 (C) 10,080 (D) 5,040 (E) None of these

7. Containers A, B, and C are partially filled with water. If we pour 3 quarts from B into A, A will contain as much water as will B and C combined. This is not done, and instead we pour 5 quarts from A into C. Now C contains as much as B, while A contains 2 quarts more than B. How many quarts of water did container A contain originally?

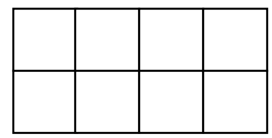
(A) 27 (B) 26 (C) 25 (D) 24 (E) 23

8. Order sin1, sin2, sin3 from smallest to largest (the angles are measured in radians).

(A) sin1 < sin2 < sin3 (B) sin3 < sin2 < sin1, (C) sin1 < sin3 < sin2,

(D) sin2 < sin1< sin3, (E) sin3 < sin1 < sin2

9. The rectangle shown is subdivided into 8 small squares. In how many ways is it possible to shade exactly one-quarter of the rectangle so that both of the following conditions are satisfied?



I. Only whole squares are shaded and
II. No two shaded squares have a common side.

(A) 18 (B) 20 (C) 24 (D) 30 (C) 36

10. Two trains take 3 seconds to clear each other when passing in opposite directions, and 35 seconds when passing in the same direction. If each train travels at a constant rate, compute the ratio of the speed of the faster train to the speed of the slower train.

(A) $\frac{19}{16}$ (B) $\frac{21}{17}$ (C) $\frac{22}{13}$ (D) $\frac{27}{19}$ (E) $\frac{31}{26}$

11. Let K be the set consisting of the first 11 positive integers. A subset of three distinct positive integers is said to be *acceptable* if it contains at most one odd element. If a subset of three different numbers is chosen at random from K, what is the probability that the subset will be *acceptable*?

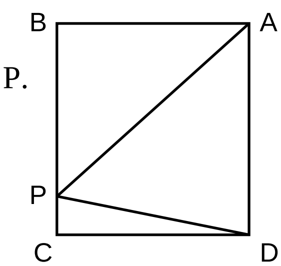
(A) $\frac{4}{11}$ (B) $\frac{9}{11}$ (C) $\frac{2}{33}$ (D) $\frac{14}{33}$ (E) $\frac{19}{33}$

12. Two numbers are written in base a as 32 and 24. The same two numbers are written in base b as 43 and 33, respectively. What is the sum of these two numbers in base 10?

(A) 33 (B) 34 (C) 35 (D) 39 (E) 41

13. A set of data consists of 12 positive integers with a mean of $6\frac{1}{3}$, a median of $6\frac{1}{2}$, and a unique mode of 7. What is the largest possible range for this set?

(A) 10 (B) 16 (C) 21 (D) 28 (E) 31



14. In rectangle ABCD, the bisector of angle BAD meets side BC at point P. The ratio of AP to PD is 4:3. What is the ratio of PD to PC.

(A) 2:1 (B) 2.5:1 (C) 3:1 (D) 3.5:1 (E) 4:1

15. For some real value of p, two of the roots of $x^3 + px^2 + 12x - 9 = 0$ have a sum of four. Compute the third root.

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $\frac{4}{3}$ (E) None of these

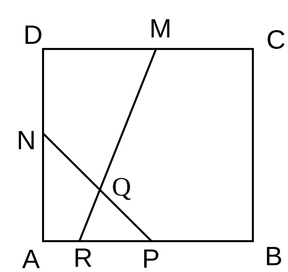
16. If the third and fourth terms of an arithmetic sequence are increased by 3 and 8 respectively, then the first four terms form a geometric sequence. Compute the third term of the arithmetic sequence.

(A) 0 (B) $\frac{3}{2}$ (C) $\frac{11}{2}$ (D) 9 (E) 12

17. If 2007! + 2008! + 2009! = (a!)(b^2), where a and b are positive integers, find the ordered pair (a,b).

(A) (2007, 2009) (B) (2008, 2009) (C) (2007, 2008)

(D) (2008, 2010) (E) (2007, 2010)



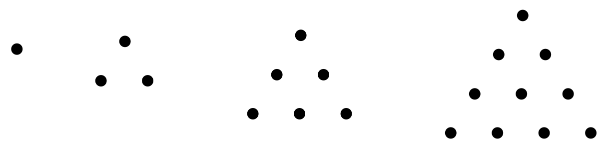
18. In the figure, M, N and P are midpoints of three sides of square ABCD, and Q is the midpoint of $\overline{NP}$. If the measure of $\angle MRP$ is x, compute $\tan x$.

(A) $\frac{3}{2}$ (B) 2 (C) $\frac{5}{2}$ (D) 3 (E) $\frac{7}{2}$

19. The degree measures of the angles of a pentagon are integral, non-equal, and form an arithmetic sequence. How many different values can the measure of the smallest angle have?

(A) 51 (B) 52 (C) 53 (D) 54 (E) 55

20. The triangular numbers, 1, 3, 6, 10, ... are so named because they can be represented geometrically as



How many of the first 100 triangular numbers end with a 0?

(A) 10 (B) 20 (C) 30 D) 40 (E) 50

21. Debbie and Don each have forty coins consisting of nickels, dimes, and quarters. Each has exactly \$5.00. If Debbie has twice as many quarters as Don, how many nickels does Don have?

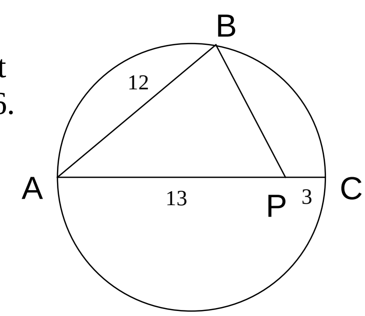
(A) 1 (B) 2 (C) 3 (D) 4 (E) None of these

22. A vertical line divides the triangle with vertices (0, 0), (1, 1), (9, 1) into two regions of equal area. The equation of the line is x = a. Compute the value of a.

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

23. In the diagram, point P is chosen on diameter $\overline{AC}$ of a circle so that it is 3 units from the circle. The diameter of the circle has length 16. Chord AB is 12 units long. Compute the distance from P to B.

(A) 5 (B) $\sqrt{65}$ (C) $\sqrt{79}$ (D) 9 (E) $\sqrt{105}$



24. The array of numbers at the right contains 13 positive integers with four of them known and nine of them unknown. Each of the four known numbers is the sum of the four unknown numbers that form a square around it (for example, $e + f + i + h = 83$). Also, e, g, b, and f are each 19 greater than a, h, i, and d respectively. Compute the sum $a + e + f + g$.

a		*b*		*c*
	77		94	
d		*e*		*f*
	86		83	
g		*h*		*i*

(A) 85 (B) 92 (C) 99 (D) 109 (E) 121

25. Let the function f be defined for all positive integers n by $f(n) = \left[\frac{n}{2}\right] - \left[\frac{n}{3}\right]$,

where $[x]$ represents the greatest integer less than or equal to x. Find the mean of all positive integers n for which f(n) = 2009.

(A) 12053.2 (B) 12053.3 (C) 12053.4 (D) 12053.5 (E) 12053.6

SOLUTIONS – KSU MATHEMATICS COMPETITION – 2009

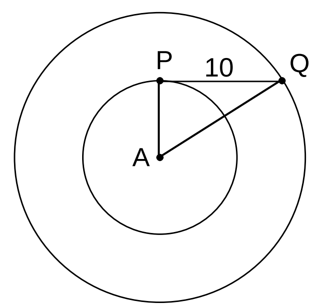
1. D Consider two lines with this property, $y = ax + a$ and $y = bx + b$. Solving $ax + a = bx + b$ gives $x = -1$ from which $y = 0$ and the intersection is **(-1, 0)**.
<u>Alternate solution</u>: Every line of the form $y = mx$ passes through (0, 0). Since $y = mx + m = m(x + 1)$ is just $y = mx$ shifted one unit to the left, the line $y = mx + m$ must pass through (-1, 0) for all values of m.

2. B Terminal zeros can only come from factors of 10. Since there is only one 2 and one 5 in the first 100 primes, and none of the other primes can have 2 or 5 as a factor, M can only have **1** terminal zero.

3. D Let E represent the number of empty seats in the room and S the number of students in class. Then $E + .16S = 13$ and $S = \frac{25(13 - E)}{4}$. Since S and E must be positive integers, the only possible pairs are (S, E) = (75, 1), (50, 5), (25, 9). Therefore, the possible values for E are 1, 5, and 9**,** the largest of which is **9**.

4. C Let $d = 100h + 10t + u - (100u + 10t + h) = 99(h - u)$. Then the difference between the number and the number with its digits reversed is a multiple of 99. Examining the three digit multiples of 99, only **594** has a units digit of 4.

5. E Draw radii AP and AQ. Using the Pythagorean Theorem on right triangle APQ, $(QA)^2 - (PA)^2 = 100$. Since the area of the large circle is $\pi(QA)^2$ and the area of the smaller circle is $\pi(PA)^2$, the area of the region between the two circles is
$$\pi(QA)^2 - \pi(PA)^2 = \pi[(QA)^2 - (PA)^2] = \mathbf{100\pi}.$$

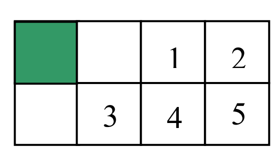


6. C There are 8 letters in **KENNESAW**, and two double repeats. Therefore, the number of arrangements is $\frac{8!}{2!2!} = \mathbf{10{,}080}$.

7. C From what we are told, we know that (i) $A + 3 = B - 3 + C$, (ii) $C + 5 = B$, and (iii) $A - 5 = B + 2$. Solving for B in (iii) $B = A - 7$. Substituting in (ii) $C = B - 5 = (A - 7) - 5 = A - 12$. Using (i) $A + 3 = A - 7 - 3 + A - 12$ or $A = \mathbf{25}$.

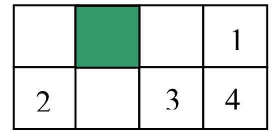
8. E An angle of 1 radian is a little less than 60°, so its sine is a bit less than sin 60°. An angle of 2 radians is a little less than 120°, so its sine is greater than sin 120° = sin 60°. An angle of 3 radians is a little less than 180°, its sine is close to 0. Therefore, $\sin 3 < \sin 1 < \sin 2$.

9. A Shading one-quarter of the rectangle means shading two squares. If the



corner square is shaded, there are five places to choose a second square as indicated in the diagram.

If a central square is shaded, four other squares can be chosen as indicated



Since each square is like one of those above, the number of possibilities for each square is

5	4	4	5
5	4	4	5

Adding these up, and remembering that each square has been counted twice, the total is **18**.

10. A Represent the rate of the faster train as R and the rate of the slower train as r. Then

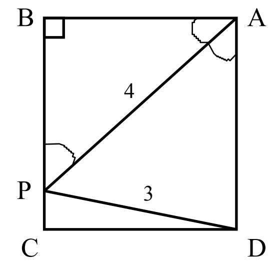
$$3R + 3r = 35R - 35r \Rightarrow 38r = 32R \Rightarrow \frac{R}{r} = \frac{38}{32} = \frac{19}{16}.$$

11. D The total number of three-element subsets is ${}_{11}C_3 = 165$. Since K contains exactly 5 even integers, the number of subsets containing no odd numbers is ${}_5C_3 = 10$, and the number of subsets with exactly one odd number is $6({}_5C_2) = 60$. Therefore, the desired probability is $\mathbf{\frac{70}{165} = \frac{14}{33}}$.

12. E $32_a = 3a + 2 = 4b + 3 = 43_b$ and $24_a = 2a + 4 = 3b + 3 = 33_b$
Therefore $3a + 2 = 4b + 3 \Rightarrow 3a - 4b = 1$ and
$2a + 4 = 3b + 3 \Rightarrow 2a - 3b = -1$
Solving together, we obtain a = 7, b = 5 and the numbers in base 10 are 23 and 18 with a sum of **41**.

13. D Since there are 12 numbers in the set, and the mean is $6\frac{1}{3}$, the sum of all the numbers is $12\left(6\frac{1}{3}\right) = 76$. Since the mean is fixed, the range can be maximized by having a large number of ones. If there were 6 ones, then 7 could not be the unique mode. If there were 5 ones and 6 sevens, the only way the median could be $6\frac{1}{2}$ is if the remaining number is a six (giving a range of 6). The combination that satisfies the requirements and leaves us free to choose the largest possible maximum is 4 ones, one 2, one 6, and 5 sevens. This set of 11 numbers has a sum of 47, leaving 76–47 = 29 as the remaining number. Thus, the required range is 29-1 = **28**.

14. C Since BC // AD, $\angle DAP \cong \angle PAB \cong \angle APB$, making ΔAPB an isosceles right triangle. Without loss of generality, let AP = 4 and PD = 3. Then, AB = CD = $2\sqrt{2}$. Using the Pythagorean Theorem on right triangle PCD, PC = 1. Therefore, the ratio of PD to PC is **3:1**.



15. C Let the roots be a, b, and c with a + b = 4. Then, abc = 9 and ab + ac + bc = 12. The second equation can be rewritten as ab + c(a+b) = 9/c + 4c = 12, or $4c^2 - 12c + 9 = (2c - 3)^2 = 0$. So, the last root is c = $\mathbf{\frac{3}{2}}$.

16. D Represent the terms of the arithmetic sequence by a, a+d, a+2d, a+3d. Then

$$\frac{a+d}{a} = \frac{a+2d+3}{a+d} \text{ and } \frac{a+d}{a} = \frac{a+3d+8}{a+2d+3}$$

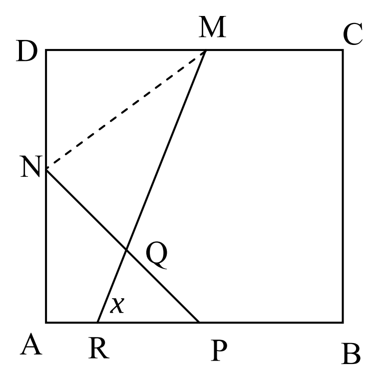
Simplifying both of these equations, we obtain $d^2 = 3a$ and $5a = 3d + 2d^2$. Solving these together for d gives d = 0, -9. If d = 0, a = 0 and the arithmetic sequence contains all zeros. Adding 3 and 8 to the third and fourth term gives 0, 0, 3, 8, which is not a geometric sequence. If d = -9, a = 27, the arithmetic sequence is 27, 18, 9, 0, and the geometric sequence is 27, 18, 12, 8. Therefore, the third term of the arithmetic sequence is **9**.

17. A The largest common factor of 2007!, 2008!, and 2009! is 2007! Therefore,
2007! + 2008! + 2009! = (2007!)[1 + 2008 + (2009)(2008)]
= (2007!)[2009 + (2009)(2008)] = (2007!)[(2009)(1 + 2008)] = $(2007!)(2009^2)$
Therefore (a, b) = **(2007, 2009)**.

18. D Construct $\overline{MN}$. Since $\angle MND = \angle PNA = 45°$, $\angle MNP$ is a right angle. Since MN = NP = 2(NQ), using right triangle MNQ, $\tan \angle MQN = \frac{MN}{NQ} = \frac{2}{1} = \mathbf{2}$. Thus $\tan\angle RQP = 2$.

Noting that $m\angle NPA = 45$, $\tan x = \tan(135 - \angle RQP) =$

$$\frac{\tan 135 - \tan\angle RQP}{1 + \tan 135 \tan\angle RQP} = \frac{-1-2}{1+(-1)(2)} = \frac{-3}{-1} = 3$$



19. C Represent the angles of the pentagon as a, a+d, a+2d, a+3d, a+4d. The formula for the sum, S, of an arithmetic sequence is S = $\frac{n}{2}[2a + (n-1)d]$. In a pentagon, S = 540.

Therefore, 540 = $\frac{5}{2}(2a + 4d)$ or a = 108 – 2d. This implies that the first term, a, will be a positive integer for all integral values of d between 0 and 54. If d = 54, we do not have a pentagon since one angle would measure 0 degrees. If d = 0, all the angles have the same measure. Therefore, there are **53** different possible measures that the smallest angle can have.

20. B The n^{th} triangular number can be represented as $\frac{n(n+1)}{2}$. For a triangular number to end in a zero, the number n(n+1) must be a multiple of 20. Therefore one of n and n+1 must be a multiple of 5 and one (maybe the same one) must be a multiple of 4. The possible values are: 4,15,19,20,24,35,39,40,44,55,59,60,64,75,79,80,84,95,99,100. Therefore, there are **20** triangular numbers that end in zero.

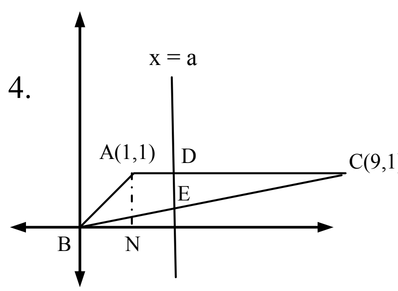
21. A For any one person, the information given can be translated into

$$5n + 10(40-n-q) + 25q = 500 \quad \text{which becomes} \quad 3q-n = 20.$$

Make a chart of all the possibilities.

n	**1**	4	7	10	13	16	19	**22**	25
d	**32**	28	24	20	16	12	8	**4**	0
q	**7**	8	9	10	11	12	13	**14**	15

Since Debbie has twice as many quarters as Don, we need to look for two columns in which the number of quarters in one is twice that in another. The two columns are in bold print. Therefore, Don has **1** nickel.

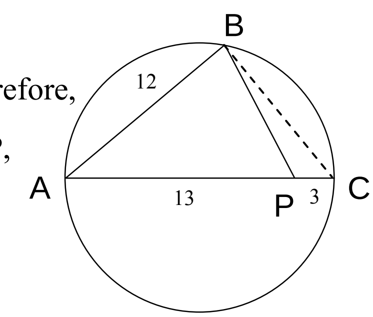


22. C Using AC as the base, the area of triangle ABC is $\frac{1}{2}(8)(1) = 4$.

Since the area of right triangle ABN = $\frac{1}{2}$, a > 1.

The equation of line BC is $y = \frac{x}{9}$. Let the coordinates of Point E be (a, $\frac{a}{9}$). Then the area of $\Delta DEC = \frac{1}{2}(1-\frac{a}{9})(9-a) = 2$.

From this, $(9-a)^2 = 36$, and a = 15 or 3. Since $a < 9$, a = 3.



23. C Draw BC. ∠ABC is a right angle (inscribed in a semicircle). Therefore, $\cos(\angle BAC) = \frac{12}{16} = \frac{3}{4}$. Using the Law of Cosines on triangle BAP,

$(BP)^2 = 12^2 + 13^2 - 2(12)(13)(\frac{3}{4}) = 79$ and $BP = \sqrt{\mathbf{79}}$.

24. C The array $\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}$ can be written as $\begin{array}{ccc} a & b & c \\ d & a+19 & d+19 \\ g & g\text{-}19 & b\text{-}19 \end{array}$

Therefore:

$$2a + b + d = 58 \quad (1)$$
$$a + b + c + d = 56 \quad (2)$$
$$a + d + 2g = 86 \quad (3)$$
$$a + b + d + g = 83 \quad (4)$$

Now express all variables in terms of b.
Subtracting (4) from (3) gives $g - b = 3$ from which $h = b - 16$.
Subtracting (2) from (4) gives $g - c = 27$ from which $c = b - 24$.
Subtracting (4) from (1) gives $a - g = -25$ from which $a = b - 22$, and $e = b - 3$.
From (1) $2b - 44 + b + d = 58$ or $d = 102 - 3b$ and then $f = 121 - 3b$.

The array $\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}$ now becomes $\begin{array}{ccc} b-22 & b & b-24 \\ 102 - 3b & b-3 & 121-3b \\ b+3 & b-16 & b-19 \end{array}$

Note, b is a positive integer and $b - 24 > 0$. Thus $b \geq 25$.
Also $102 - 3b > 0$. Thus $b \leq 33$. Therefore, $b = 25, 26, 27, \ldots, 33$.
So nine different arrays can be formed. For any of them,
$a + e + f + g = (b - 22) + (b - 3) + (121 - 3b) + (b + 3) =$ **99.**

25. D We are given $f(n) = \left[\frac{n}{2}\right] - \left[\frac{n}{3}\right]$. Suppose f(n) = k>0. If n = 6k,

$$f(n) = \left[\frac{6k}{2}\right] - \left[\frac{6k}{3}\right] = [3k] - [2k] = 3k - 2k = k.$$

A place to look for other values is at or near 6k.

Try n = 6k+1: $f(6k+1) = \left[\frac{6k+1}{2}\right] - \left[\frac{6k+1}{3}\right] = \left[\frac{3k+1}{2}\right] - \left[\frac{2k+1}{3}\right] = 3k - 2k = k.$

However, $f(6k+2) = \left[\frac{6k+2}{2}\right] - \left[\frac{6k+2}{3}\right] = [3k+1] - \left[\frac{2k+2}{3}\right] = 3k + 1 - 2k = k + 1$

Similar (careful) calculations yield the following results: f(6k+3) = k, f(6k+4) = k+1, f(6k+5) = k+1, f(6k-1) = k, f(6k-2) = k, f(6k-3) = k-1, f(6k-4) = k, f(6k-5) = k-1.
It is easy to verify that $f(n) \neq k$ for any n greater than 6k+5 or less than 6k-5.
Therefore the solutions are 6(2009)-4, 6(2009)-2, 6(2009)-1, 6(2009), 6(2009)+1, and 6(2009)+3. The mean of these is 6(2009) - .5 = **12053.5**.