THE 2011–2012 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

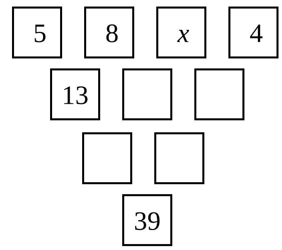
PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box, or more than one box, is marked. Zero points are given for an incorrect answer. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. In the puzzle at the right, the number in each empty square is obtained by adding the two numbers in the row directly above. For example, 5 + 8 = 13. What is the value of x?



(A) 2 (B) 3 (C) 6 (D) 7 (E) 9

2. A circle passes through the points (0, 0), (0, 2) and (4, 0). What is the area of this circle?

(A) 5π (B) 8π (C) 9π (D) 10π (E) 16π

3. Tom found the value of 3^{21} = 10,4A0,353,20B. He found all the digits correctly except the fourth and last digits, denoted by A and B, respectively. What is the value of A?

(A) 0 (B) 2 (C) 3 (D) 6 (E) 8

4. In determining standings in a certain hockey league, a team receives 3 points for each win, 1 point for each tie, and –1 point for each loss. After playing 50 games, the Ducks have a total of 76 points. How many more wins than losses do the Ducks have at this time in the season?

(A) 12 (B) 13 (C) 14 (D) 15 (E) 16

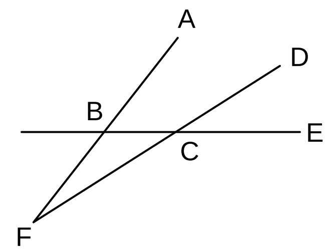
5. Let x = m + n where m and n are positive integers satisfying $2^6 + m^n = 2^7$. The sum of all of the possible values of x is:

(A) 18 (B) 25 (C) 75 (D) 82 (E) 90

6. Call an integer "cwazy" if it is divisible by each of its digits. For example, 36 is "cwazy" because it is divisible by both 3 and 6. However, 28 and 71 are not "cwazy". How many "cwazy" integers are there between 10 and 100?

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

7. If the measure of $\angle$ABE is 6 degrees greater than the measure of $\angle$DCE, compute the number of degrees in the measure of $\angle$AFD.



(A) 6° (B) 8° (C) 10° (D) 12° (E) 16°

8. If q and r are the zeros of the quadratic polynomial $x^2 + 15x + 31$, find the quadratic polynomial whose zeros are $q + 1$ and $r + 1$.

(A) $x^2 + 17x + 31$ (B) $x^2 + 15x + 33$ (C) $x^2 + 13x + 17$

(D) $x^2 + 19x + 37$ (E) None of these

9. The makers of Delight Ice Cream put a coupon for a free ice cream bar in every 80^{th} bar they make. They put a coupon for 2 free bars in every 180^{th} bar and a coupon for 3 free bars in every 300^{th} bar. If they put all three coupons in every n^{th} bar, compute n.

(A) 1200 (B) 1800 (C) 2400 (D) 3600 (E) 5400

10. Starting at opposite ends of a straight moving walkway at an airport, which travels at a constant rate of k ft/sec, Don and Debbie walk towards each other (Don moving in the direction the walkway is moving, Debbie moving against the direction the walkway is moving). They meet at a point one-seventh of the way from one end of the walkway. If they were on a normal (non-moving) floor they would each walk at a rate of 3 feet per second. Determine the value of k.

(A) $\frac{12}{7}$ (B) $\frac{13}{9}$ (C) $\frac{15}{7}$ (D) $\frac{15}{9}$ (E) $\frac{17}{9}$

11. In triangle ABC, AB = 3, BC = 5, and AC = 7. The angle bisectors of the two smallest angles of the triangle meet at point P. Compute the ratio of the measure of $\angle$APC to the measure of $\angle$B?

(A) $\frac{11}{10}$ (B) $\frac{10}{9}$ (C) $\frac{8}{7}$ (D) $\frac{7}{6}$ (E) $\frac{5}{4}$

12. During a vacation it rained on 13 days, but when it rained in the morning the afternoon was sunny, and every rainy afternoon was preceded by a sunny morning. There were 11 sunny mornings and 12 sunny afternoons. How many days long was the vacation?

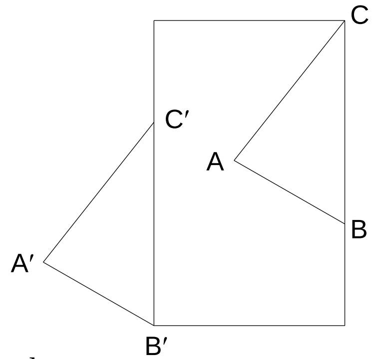
(A) 16 (B) 17 (C) 18 (D) 19 (E) 23

13. The lengths of two sides of a triangle are $\log_2 4$ and $\log_4 2$. If the length of the third side is $\log_3 x$, which of the following is NOT a possible value for x?

(A) 5 (B) 7 (C) 9 (D) 11 (E) They are all possible

14. If $\cos A - \cos B = -\sin 80°$ and $A + B = 60°$ for $0° < A < 180°$, determine the largest possible value of A.

(A) 20° (B) 80° (C) 110° (D) 130° (E) 160°



15. In a 15 x 24 rectangle, congruent triangles ABC and A′B′C′ are drawn as shown, with corresponding sides parallel. If AB = 12, AC = 14, and BC = 16, what is the distance from A′ to A?

(A) 15 (B) 16 (C) 17 (D) 18 (E) 19

16. If $\frac{4}{2011} < \frac{a}{a+b} < \frac{5}{2011}$, compute the number of integral values of $\frac{b}{a}$.

(A) 20 (B) 40 (C) 60 (D) 80 (E) 100

17. On Dr. Garner's multiple choice tests, each question has 4 choices, exactly one of which is the correct answer. On one such test, a student knows the correct answer to exactly 70% of the questions. For the other 30% of the questions he selects one of the choices at random. If he gets the correct answer to a particular question, what is the probability that he knew the answer rather than guessed it?

(A) $\frac{3}{40}$ (B) $\frac{7}{20}$ (C) $\frac{17}{40}$ (D) $\frac{31}{40}$ (E) $\frac{28}{31}$

18. For all real numbers x, the function $f(x)$ satisfies $2f(x) + f(1-x) = x^2$. What is the value of $f(5)$?

(A) $\frac{15}{4}$ (B) $\frac{21}{5}$ (C) $\frac{25}{3}$ (D) $\frac{27}{4}$ (E) $\frac{34}{3}$

19. From the point $P(3, 0)$ a tangent line with positive slope is drawn to the graph of $\frac{x^2}{4} + y^2 = 1$. Determine the slope of the tangent line.

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{3}$ (D) $\frac{1}{5}$ (E) $\frac{\sqrt{5}}{5}$

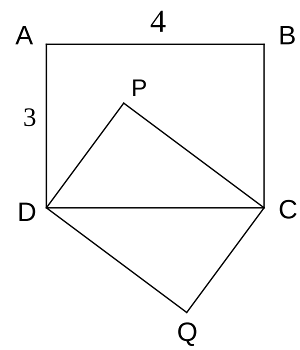
20. The number 2011 can be written as $a^2 - b^2$ where a and b are integers. Compute the value of $a^2 + b^2$.

(A) 2018041 (B) 2022061 (C) 2024072 (D) 2026085 (E) 2033051

21. Consider the following system of equations: (1) ax + by = c and (2) dx + ey = f ($c \neq 0$, $f \neq 0$). When x = 0, equation (1) yields y = 3 and (2) yields y = 6. When y = 0, (1) yields x = -3 and (2) yields x = 3. What is the common solution (x, y) for the system?

(A) (1, 2) (B) (2, 6) (C) (4, 1) (D) (6, 2) (E) (1, 4)

22. Rectangle ABCD has sides of length 3 and 4. Rectangle PCQD is similar to rectangle ABCD, with P inside rectangle ABCD. Compute the distance from P to AB.



(A) 1 (B) $\frac{4}{3}$ (C) $\frac{7}{5}$ (D) $\frac{21}{17}$ (E) $\frac{27}{25}$

23. Let S(n) = $[\sqrt{1}] + [\sqrt{2}] + \ldots + [\sqrt{n}]$ where [k] is the greatest integer less than or equal to k. Compute the largest value of k < 2011 such that S(2011) – S(k) is a perfect square.

(A) 1997 (B) 1998 (C) 1999 (D) 2000 (E) 2001

24. Anne writes down the thirteen consecutive integers from −6 to 6. She then performs a series of operations. In each operation she identifies two numbers that differ by two, decreases the larger by one, and increases the smaller by one so that the two numbers are now equal. After a while she has thirteen zeros left and stops. How many operations did she perform?

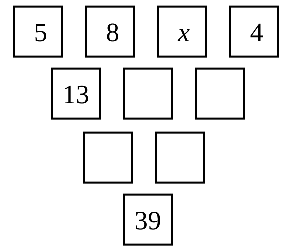
(A) 78 (B) 84 (C) 91 (D) 95 (E) 99

25. Isosceles triangle ABC has sides which measure 20, 20, and 24. M, N, and P are the midpoints of the sides and folds are made along MN, MP, and NP. A tetrahedron is formed when A, B, and C are made to coincide. If the volume of the tetrahedron formed is $a\sqrt{b}$, where b is a prime number, find the ordered pair (a,b).

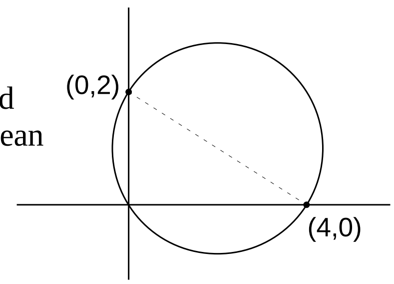
(A) (48, 2) (B) (48, 3) (C) (48, 7) (D) (54, 3) (E) (54, 5)

THE 2011–2012 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

SOLUTIONS



1. A The entries in the four empty boxes are, from top to bottom and left to right, $x + 8$, $x + 4$, $x + 21$ and $2x + 12$. Then $(x + 21) + (2x + 12) = 39$ or $3x + 33 = 39$, and $x = 2$.



2. A Because the angle at (0, 0) is a right angle, it is inscribed in a semicircle, which makes the segment connecting (0, 2) and (4, 0) a diameter. Using the distance formula (or the Pythagorean Theorem), the diameter of the circle is $\sqrt{20}$, making the area $\frac{1}{4}(20\pi) = 5\pi$.

3. D Of course, one could compute the value of 3^{21} directly, but that would take some time and might lead to careless errors. A more general approach is as follows. Let's find B first. The powers of 3, taken in order from 3^1, end in the repeating pattern 3, 9, 7, 1. Since 21 is one more than a multiple of 4, B = 3. Since 3^{21} is a multiple of 9, its digits must sum to a multiple of 9. Since the known digits and B have a sum of 21, the missing digit A must be 6.

4. B Let W = the number of wins, T = the number of ties, and L = the number of losses. W + T + L = 50 and 3W + T - L = 76. Subtracting the first equation from the second gives 2W - 2L = 26 and W - L = **13**.

5. E If $2^6 + m^n = 2^7 = 2(2^6)$, then $m^n = 2^6$. Further,

$$2^6 = (2^2)^3 = (2^3)^2 = (2^6)^1$$

Thus, the possible values for m and n are

$m = 2; n = 6 \Rightarrow x = 8$

$m = 2^2 = 4; n = 3 \Rightarrow x = 7$

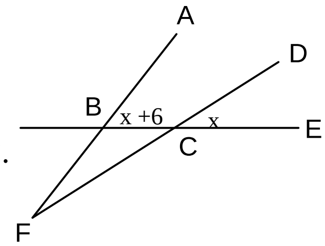
$m = 2^3 = 8; n = 2 \Rightarrow x = 10$

$m = 2^6 = 64; n = 1 \Rightarrow x = 65$

Thus, the sum of the possible values for x is 8 + 7 + 10 + 65 = 90.

6. D The numbers are 11, 22, 33, 44, 55, 66, 77, 88, 99, 12, 24, 36, 48, and 15. Trial and error will work but takes some time. However, writing a two digit integer as 10a + b and using the definition you find that only numbers with b = a, b = 2a, and b = 5a work. Therefore, there are 14 cwazy numbers between 10 and 100.

7. A Represent m$\angle$FBC as 180 – (x + 6) = 174 – x. Also, m$\angle$DCE = m$\angle$BCF = x. Then m$\angle$AFD = 180 – (174 –x) – x = 6°.



8. C We could find the zeros of the given polynomial, increase each by 1, and use them to find the answer. However, that is time consuming. Here are two shorter methods.

Method 1: From Vieta's Theorem, we have q + r = –15 and qr = 31. Then, (q + 1) + (r + 1) = q + r + 2 = –13 and (q+1)(r+1) = qr + q + r + 1 = 31 – 15 + 1 = 17. Apply Vieta's Theorem again to get the desired polynomial is **$x^2 + 13x + 17$**.

Method 2: One can imagine that increasing the roots both by 1 is the result of shifting the graph of y = $x^2 + 15x + 31$ rightward by 1. Thus, the desired polynomial is $(x-1)^2 + 15(x-1) + 31 = x^2 - 2x + 1 + 15x - 15 + 31 =$ **$x^2 + 13x + 17$**.

9. D Clearly, we need the least common multiple of 80, 180, and 300. $80 = (2^4)(5)$, $180 = (2^2)(3^2)(5)$, and $300 = (2^2)(3)(5^2)$. The LCM is $(2^4)(3^2)(5^2) =$ **3600**.

10. C If Don is walking in the direction the walkway is moving, his rate is 3 + k while Debbie's is 3 – k. Meeting at a point 1/7 from the end means that Don has walked six times as far. If t represents the time it takes for them to meet,

$(3 + k)t = 6(3 - k)t$. Since $t \neq 0$, $3 + k = 18 - 6k$ making $k = \frac{15}{7}$.

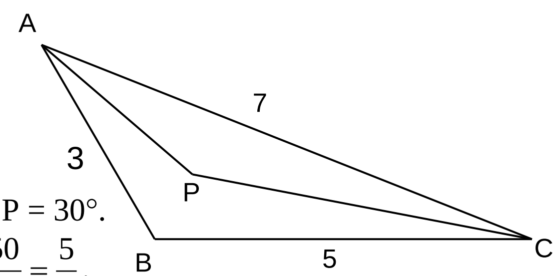
11. E Using the Law of Cosines on ΔABC,

$7^2 = 3^2 + 5^2 - 2(3)(5)\cos B$ from which

$\cos B = -\frac{1}{2}$ and m$\angle$B = 120°.

Since m$\angle$ABC + m$\angle$ACB = 60°, m$\angle$PAC + m$\angle$ACP = 30°.

Therefore, m$\angle$P = 150°. Thus the desired ratio is $\frac{150}{120} = \frac{5}{4}$.



12. C Only three kinds of days are possible: (sunny, rainy), (rainy, sunny) or (sunny, sunny). Let a = the number of (sunny, rainy), b = the number of (rainy, sunny), and c = the number of (sunny, sunny). Then,

$$\begin{aligned} a + b &= 13 \\ \left.\begin{aligned} a + c &= 11 \\ b + c &= 12 \end{aligned}\right\} \end{aligned} \quad \left.\begin{aligned} a + b &= 13 \\ a - b &= -1 \end{aligned}\right\} \quad 2a = 12 \Rightarrow \begin{aligned} a &= 6 \\ b &= 7 \\ c &= 5 \end{aligned}$$

Length of vacation was 6 + 7 + 5 = **18** days.

13. A $\text{Log}_2 4 = 2$ and $\log_4 2 = \frac{1}{2}$. Using the triangle inequality, we have

$$\log_3 x < 2 + \frac{1}{2} \quad \text{and} \quad \log_3 x + \frac{1}{2} > 2$$

Therefore, $\log_3 x < \frac{5}{2} \Rightarrow x < 3^{\frac{5}{2}}$ or $x < 9\sqrt{3}$ and

$$\log_3 x + \frac{1}{2} > 2 \Rightarrow \log_3 x > \frac{3}{2} \text{ or } x > 3\sqrt{3}.$$

Therefore, the set of all possible values of x is $3\sqrt{3} < \mathbf{x} < 9\sqrt{3}$. Since $3\sqrt{3} \approx 5.2$ and $9\sqrt{3} \approx 15.6$, choice A (5) is the only choice that is not possible.

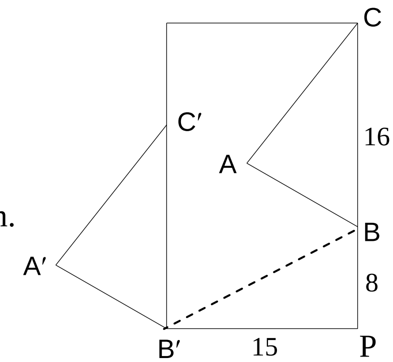
14. D Given cosA – cosB = –sin80, and A+ B = 60, we obtain

cosA – cos(60 – A) = cosA – (cos60cosA + sin60sinA) =

$\cos A - \frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A = -\frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A = -\sin 80.$

Thus, $\frac{1}{2}\cos A + \frac{\sqrt{3}}{2}\sin A$ = (sinA)(cos30) – (cosA)(sin30) = sin(A – 30) = sin80, making A = 110 or 130. The larger of the two is 130°.

15. C Constructing BB′, and using the Pythagorean Theorem on triangle BPB′, we get BB′ = 17. Since AB and A′B′ are parallel and congruent, quadrilateral ABB′A′ is a parallelogram. Therefore, BB′ = AA′ = **17**.



16. E $\frac{4}{2011} < \frac{a}{a+b} < \frac{5}{2011} \Rightarrow \frac{2011}{5} < \frac{a+b}{a} < \frac{2011}{4} \Rightarrow \frac{2011}{5} < 1 + \frac{b}{a} < \frac{2011}{4}$.

Therefore, $401\frac{1}{5} < \frac{b}{a} < 501\frac{3}{4}$, which means there are **100** integral values of $\frac{b}{a}$.

17. E We are given: P(student knows answer) = $\frac{7}{10}$ and P(doesn't know answer) = $\frac{3}{10}$.

We also know that when a student guesses, P(guessing correctly) = $\frac{1}{4}$.

Therefore, P(doesn't know answer and guesses correctly) = $\frac{3}{10} \cdot \frac{1}{4} = \frac{3}{40}$.

The overall probability of getting a correct answer is $\frac{7}{10} + \frac{3}{40} = \frac{31}{40}$. Therefore, the probability that the student knew the answer to the question is $\frac{7}{10} \div \frac{31}{40} = \frac{28}{31}$.

18. E Using x = 5, we obtain $2f(5) + f(-4) = 25$.
Using x = –4, we obtain $2f(-4) + f(5) = 16$.
Multiplying the first equation by 2 and subtracting the equations we obtain
$-3f(5) = -34$ from which $f(5) = \mathbf{\frac{34}{3}}$.

19. E Let m be the slope of the line tangent to the ellipse. The equation of the tangent line is y = m(x – 3). Substituting into the equation of the ellipse,

$$\frac{x^2}{4} + (mx - 3m)^2 = 1 \quad \Rightarrow \quad (4m^2 + 1)x^2 - 24m^2 x + 4(9m^2 - 1) = 0.$$

For the line to be tangent, this equation must have exactly one solution. Setting the discriminant equal to zero, we obtain

$(24m^2)^2 - 16(4m^2 + 1)(9m^2 - 1) = 0 \quad \Rightarrow \quad 5m^2 = 1$. Thus, m = $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

20. B Method 1: We are given that $a^2 - b^2 = (a - b)(a + b) = 2011$. Since 2011 is a prime number, this means a – b = 1 and a + b = 2011. Solving these two equations together, a = 1006 and b = 1005. Thus $a^2 + b^2 = 1006^2 + 1005^2 = 2022061$.

Method 2: Any odd number can be expressed as the difference between the squares of two consecutive integers. Therefore, $(x + 1)^2 - x^2 = 2011$. Thus, 2x + 1 = 2011 and x = 1005. Setting a = 1006 and b = 1005, $a^2 + b^2 = 1006^2 + 1005^2 = 2022061$.

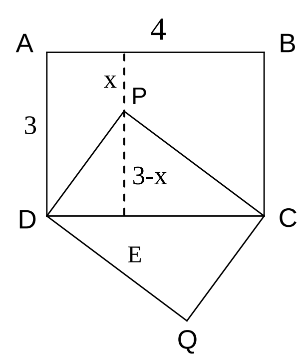
21. E Using x = 0, from (1) $3b = c \Rightarrow b = \frac{c}{3}$ and from (2) $6e = f \Rightarrow e = \frac{f}{6}$.

Using y = 0, from (1) $-3a = c \Rightarrow a = -\frac{c}{3}$ and from (2) $3d = f \Rightarrow d = \frac{f}{3}$.

Substituting in the two given equations (1) becomes $-\frac{c}{3}x + \frac{c}{3}y = c$ or

-x + y = 3 and (2) becomes $\frac{f}{3}x + \frac{f}{6}y = f$ or 2x + y = 6. Solving simultaneously for x and y gives x = 1 and y = 4 or **(1, 4)**.

22. E Because the diagonal of rectangle ABCD is 5, compute the length of PD as follows $\frac{3}{PD} = \frac{5}{4}$ making PD = $\frac{12}{5}$. Similarly, PC = $\frac{16}{5}$.
Draw the perpendicular through P to AB and CD, as shown.



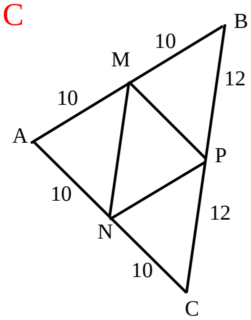
Using similar right triangles DPE and DCP, $\frac{DP}{DC} = \frac{PE}{PC}$ from which

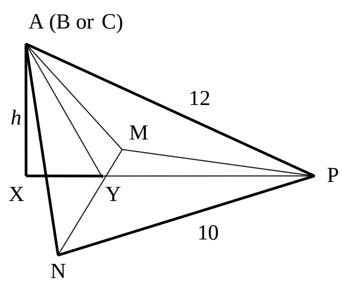
(DC)(PE) = (DP)(PC). Thus, $4(3-x) = \frac{12}{5}\cdot\frac{16}{5}$ and x = $\mathbf{\frac{27}{25}}$.

23. D Since $\sqrt{1936}$ = 44 and $\sqrt{2025}$ = 45, all numbers from [$\sqrt{1936}$] to [$\sqrt{2011}$] must equal 44. If $K \geq 1936$, S(2011) - S(k) = [$\sqrt{2011}$] + [$\sqrt{2010}$] + **…** + [$\sqrt{K+1}$] = 44(2011 - K) = (4)(11)(2011 - K). Therefore, S(2011) - S(K) will be a perfect square for 2011 - K = 11, and K = **2000**.

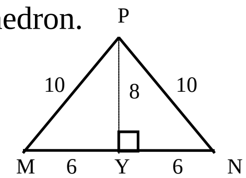
24. C Call the two numbers that Anne changes during an operation a+1 and a−1. Before the operation, the sum of the squares of these two is $a^2+2a+1+a^2-2a+1$ = $2a^2 + 2$, whereas after the operation the sum of the squares is simply $2a^2$. Since the other numbers do not change, we see that the sum of the squares of all nine numbers goes down by two during every operation. Because
$[(-6)^2+(-5)^2+(-4)^2+(-3)^2+(-2)^2+(-1)^2+0^2+1^2+2^2+3^2+4^2+5^2+6^2]$
$-(0^2+0^2+0^2+0^2+0^2+0^2+0^2+0^2+0^2) = 182$, Anne performed 182÷2 = 91 operations.

25. C

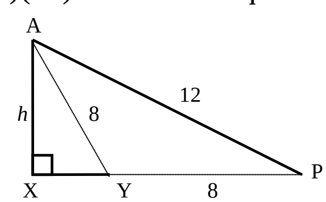




We need to find the area B of triangular base MNP (whose sides are 10, 10, and 12) and the length of *h*, the altitude of the tetrahedron.



It is easy to see that the area of triangle MNP is ½(8)(12) or B = 48 square units



Next we find the length of *h*. In the middle diagram above, triangle PYA has sides PA = 12, PY = 8 and AY = 8. Thus, triangle PYA is obtuse, as shown. Using the Law of Cosines on triangle PYA.

$$144 = 64 + 64 - 128\cos(\angle APY) \quad \text{and} \quad \cos(\angle AYP) = -\frac{1}{8}$$

Therefore, $\cos(\angle AYX) = \frac{1}{8}$, making XY = 1 and $h = 3\sqrt{7}$. Hence, V = $\frac{1}{3}Bh = \frac{1}{3}(48)(3\sqrt{7}) = 48\sqrt{7}$

The desired ordered pair is **(48, 7)**.