THE 2011-2012 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

Calculators are <u>NOT</u> permitted

1. Find, with proof, all integers n such that $2^6 + 2^9 + 2^n$ is the square of an integer.

2. Find, with proof, all real numbers a such that $|x-1| - |x-2| + |x-4| = a$ has <u>exactly</u> 3 solutions.

3. As Lisa hurried to copy down the last problem of her math homework assignment at the end of class, she got as far as

$$0 = 9x^8 - 28x^6 -$$

when her teacher abruptly erased the board and ended class. Looking at her notes later that evening, Lisa could recall only that the polynomial in the equation had rational coefficients, and was written with decreasing exponents, all of which were non-negative integers. She remembered that the teacher told the class to compute the one root of this equation that was unequal to the other seven identical roots. Help Lisa by finding this one root and include for her a proof.

4. A *pair-square* set of size n is a set of n distinct positive integers such that each pair of integers in the set has a sum which is the square of an integer. For example, the set {15, 34, 66} is a *pair-square* set of size 3 because $15 + 34 = 7^2$, $34 + 66 = 10^2$, and $15 + 66 = 9^2$.

   a. Find a *pair-square* set of size 3 containing the number 2012.

   b. Prove that any *pair-square* set of size 3 contains at most one odd number.

5. The length of the shorter side of a parallelogram is 2012. The bisectors of the two acute interior angles are 20 units apart. The bisectors of the two obtuse interior angles are 21 units apart. Compute, with proof, the length of the longer side of the parallelogram.

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PART II

SOLUTIONS

1. If $m^2 = 2^6 + 2^9 + 2^n = 576 + 2^n = 24^2 + 2^n$, then $2^n = m^2 - 24^2 = (m-24)(m+24)$. Therefore, m + 24 and m – 24 must each be powers of 2. Let $m + 24 = 2^k$ and $m - 24 = 2^p$ where, p < k and p + k = n. Then $2^k - 2^p = 48$ which implies 2^p divides 48, so that $p \le 4$. Trying p = 0, 1, 2, 3, 4 gives p = 4, k = 6 and n = 10 as the only possible value for n.

2. We begin by graphing the function $y = f(x) = |x-1| - |x-2| + |x-4|$. If $x \le 1$, then $x - 1 \le 0$, $x - 2 \le 0$ and $x - 4 \le 0$, so we have

$$y = -(x-1) + (x-2) - (x-4) = -x + 3.$$

If $1 \le x \le 2$, then then $x - 1 \ge 0$, $x - 2 \le 0$ and $x - 4 \le 0$, so

$$y = (x-1) + (x-2) - (x-4) = x + 1.$$

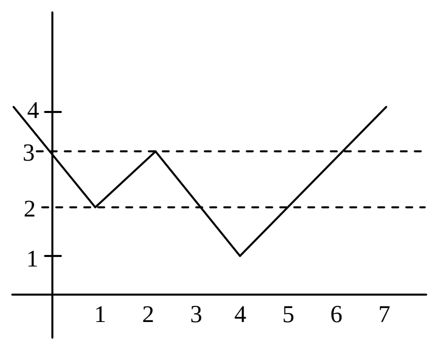
In the interval $2 \le x \le 4$ we have $x - 1 \ge 0$, $x - 2 \ge 0$ but $x - 4 \le 0$, so

$$y = (x-1) - (x-2) - (x-4) = -x + 5.$$

Finally, for $x \ge 4$ we have

$$y = (x-1) - (x-2) + (x-4) = x - 3.$$

By piecing together the relevant parts of these four linear functions, we get the graph of the function f(x) shown at the right.



Thus, for the original equation to have exactly three solutions, we have to choose *a* so that the horizontal line y = *a* touches the graph of f(x) at exactly three points. From the graph, this happens only for *a* = 2 and *a* = 3.

3. $P(x) = 9x^8 - 28x^6 - \ldots$ Since the coefficient of x^7 is zero, the sum of the roots of P(x) = 0 is zero. Let the seven identical roots be *a* and the desired eighth root be *b*. Then 7*a* + *b* = 0 or *b* = -7*a*. Standardizing P(x), the sum of the products of the roots taken two at a time is $-\frac{28}{9}$. Therefore, ${}_7C_2 a^2 + 7ab = -\frac{28}{9}$ or $21a^2 + 7ab = -\frac{28}{9}$.

Substituting -7*a* for *b* we obtain $a = \pm\frac{1}{3}$. Hence there are two possible pairs of values for *a* and *b*: $a = \frac{1}{3}, b = -\frac{7}{3}$ or $a = -\frac{1}{3}, b = \frac{7}{3}$. However, by Descartes' rule of signs, there are at most six positive roots of P(x), so the value of *a* cannot be $\frac{1}{3}$. Therefore, the value of *b* is $\frac{7}{3}$

4. a. Let the desired *pair-square* set be {2012, a, b}. Since 2012 + 13 = 2025 = 45^2, try a = 13 as a second member of the set. Then

$$2012 + b = x^2 \quad \text{and} \quad 13 + b = y^2 \text{ for some integers x and y.}$$

Subtracting these two equations gives $1999 = x^2 - y^2 = (x + y)(x - y)$.
Since 1999 is a prime number, x + y = 1999 and x – y = 1. From these two equations we obtain x = 1000. Then $b = 1000^2 - 2012 = 1000000 - 2012 = 997988$.
Thus, one desired *pair-square* set of size 3 is {13, 2012, 997988}.

(Note: {292, 2012, 45077} and {488, 2012, 143912} also work. There are others.)

b. Every integer has one of the four forms 4k; 4k + 1; 4k + 2 and 4k + 3 for integers k.

First we prove that the square of an integer must have one of the forms 4n or 4n + 1.

Proof:

(i) $(4k)^2 = 4(4k^2) = 4n$

(ii) $(4k+1)^2 = 16k^2 + 8k + 1 = 4(4k \quad + 2k) + 1 = 4n + 1$

(iii) $(4k+2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1) = 4n$

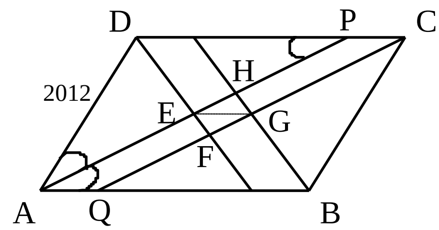
(iv) $(4k+3)^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1 = 4n + 1$

Therefore, the square of an integer must have one of the forms 4n or 4n + 1.

Let S be a *pair-square* set of size 3. Suppose that the set S contains the two odd numbers a and b. Since $a + b$ is an even square, it must have form 4n, and therefore a and b cannot both have form 4k + 1, nor can they both have form 4k + 3. It follows that we can write $a = 4k + 1$ and $b = 4k + 3$.

We derive a contradiction by showing that there is no possibility for the third member z of S. Indeed, if z has form 4k or 4k + 3, then $z + b$ is not a square, and if z has form 4k + 1 or 4k + 2, then $z + a$ is a non-square. It follows that S can have no more than one odd number.

5. It is easy to show that ΔADP is isosceles (note the marked congruent angles).



Thus DP = 2012 and all we need is the length of PC.
Since ΔADP and ΔCBQ are congruent isosceles triangles, $\angle PAB \cong \angle DAP \cong \angle CQB$, making $\overline{AP}$ parallel to $\overline{QC}$.
Similarly, the other two angle bisectors are parallel, making EFGH a parallelogram.

Since $\angle DAB$ and $\angle ADC$ are supplementary, $m\angle DAB + m\angle ADC = 180$.
Since $m\angle DAE = \frac{1}{2}\, m\angle DAB$ and $m\angle ADE = \frac{1}{2}\, m\angle ADC$
we know $m\angle DAE + m\angle ADE = \frac{1}{2}(m\angle DAB + m\angle ADC) = 90$.
Therefore, ΔADE is a right triangle and $m\angle DEA = m\angle HEF = 90°$, making EFGH a rectangle. This means that EF = 20 and FG = 21. Using the Pythagorean Theorem on triangle EFG, EG = 29. Since $\Delta DPE \cong \Delta BCG$ (AAS) we know $\overline{PE} \cong \overline{CG}$. Since they are also parallel, EGCP is a parallelogram, so that EG = PC = 29. Therefore, the length of the longer side of ABCD is 2012 + 29 = **2041**.