THE 2013–2014 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box, or more than one box, is marked. Zero points are given for an incorrect answer. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. Six bags of marbles contain 21, 22, 26, 27, 32, and 35 marbles, respectively. All the marbles in one bag are chipped. The other five bags contain no chipped marbles. Debbie takes three of the bags and Don takes two of the others. Only the bag of chipped marbles remains. If Debbie gets twice as many marbles as Don, how many marbles are chipped?

   (A) 22 (B) 26 (C) 27 (D) 32 (E) None of these

2. The integers greater than one are listed in five columns as shown. In which column will the number 2013 fall?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Column	1	2	3	4	5
		2	3	4	5
	9	8	7	6	
		10	11	12	13
	17	16	15	14	

3. In a group of men and women, the average age is 31 years. If the average age of the men is 35 years and the average age of the women is 25 years, what is the ratio of the number of men to the number of women?

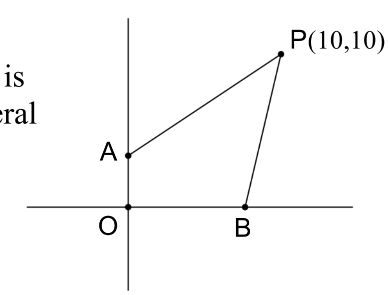
   (A) $\frac{2}{1}$ (B) $\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $\frac{4}{3}$ (E) $\frac{5}{2}$

4. If X, Y, and Z are distinct positive integers such that $\dfrac{X^3+Y^3}{X^3+Z^3}=\dfrac{X+Y}{X+Z}$, then X must equal which of the following?

   (A) $Y+Z$ (B) Y^2+Z^2 (C) $(Y+Z)^2$ (D) YZ (E) $2YZ$

5. In the graph shown, point P has coordinates (10, 10), and point B is twice as far from the origin as is point A. If the area of quadrilateral PAOB is 30, compute the y-coordinate of point A.

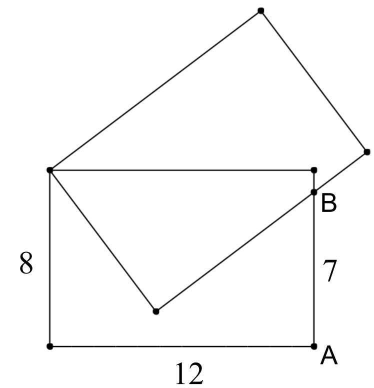
   (A) 1.5 (B) 2 (C) 2.25 (D) 2.5 (E) 3



6. Of the 50 states, there are exactly twice as many larger than Hawaii as there are larger than Florida. There are exactly four times as many states smaller than Florida as there are states smaller than Hawaii. How many states are both larger than Hawaii and smaller than Florida?

(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

7. Two 8 x 12 rectangles share a common corner and overlap as in the diagram, so that the distance, AB, from the bottom right corner of one rectangle to the intersection point B along the right edge of that rectangle is 7. What is the area of the region common to the two rectangles?



(A) 36 (B) 38 (C) 40 (D) 42 (E) 44

8. If the three digit number $\underline{A}\ \underline{B}\ \underline{C}$ is decreased by the sum of its digits, the result is a perfect square. Which of the following is not a possible value for A?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

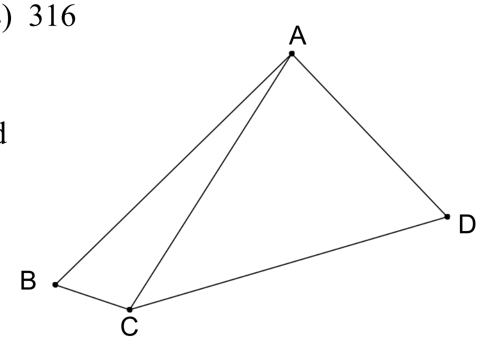
9. Which of the following could not be the discriminant of the quadratic equation $ax^2 + bx + c = 0$, where a > 0 and a, b, and c are all integers?

(A) 85 (B) 99 (C) 117 (D) 121 (E) 133

10. Let $N = 2^a + 2^b + 2^c$, where a, b, and c are distinct positive integers all greater than 100. Which of the following is the smallest sum $a + b + c$ such that N is a perfect square.

(A) 307 (B) 308 (C) 310 (D) 313 (E) 316

11. In the diagram, $\overline{AB} \cong \overline{CD}$, m∠ADC = 63°, m∠DCA = 41°, and m∠ACB = 104°. Compute the measure of ∠BAC.



(A) 11° (B) 13° (C) 15° (D) 17° (E) 19°

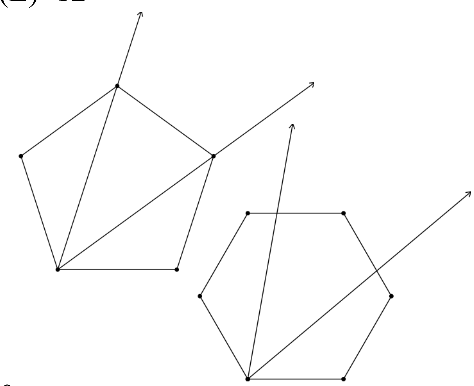
12. Let $x = 3^a + 3^b$. If a and b are chosen independently from the integers 1 to 100, inclusive, and are equally likely to be chosen, compute the probability that x is a multiple of 5.

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

13. In Dr. Garner's math class, students' grades are based exclusively on tests, all of which have equal weight. When one student got a grade of 98 on a certain test, it raised her average by 1 point. But when she got a grade of 70 on the next test, her average then dropped by 2 points. Including these two tests, how many tests did she take?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

14. Regular pentagons have an interesting property. The angle trisectors of any angle of a regular pentagon intersect other vertices of the pentagon; i.e. they contain diagonals of the pentagon. Regular hexagons do not have this property (see the two illustrations at the right). Which of the following represents the number of sides of a regular polygon that also has this property?



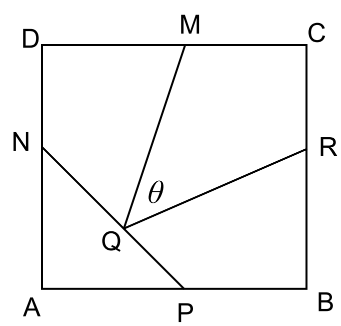
(A) 70 (B) 75 (C) 80 (D) 85 (E) 90

15. If $i + 2i^2 + 3i^3 + 4i^4 + \ldots + 2013i^{2013} = A + Bi$, where A and B are real numbers and i is the imaginary unit, compute the ordered pair (A, B).

(A) (1006, 1007) (B) (1006, -1007) (C) (-1006, -1007)

(D) (2012, 2013) (E) (2012, -2013)

16. In the figure, M, N, P and R are midpoints of the sides of square ABCD, and Q is the midpoint of $\overline{NP}$. If the measure of $\angle MQR = \theta$, compute $\tan\theta$.



(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{3}{2}$ (E) $\frac{4}{3}$

17. The function f has the following properties:

(i) $f(1) = 1$

(ii) $f(n) = n + f(n-1)$ for all integers $n \geq 2$.

Compute $f(2013)$.

(A) 2193052 (B) 2027091 (C) 1012538 (D) 506269 (E) 142013

18. Equilateral triangles ABC and ABD lie on perpendicular planes. Compute the cosine of angle CAD.

(A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{\sqrt{2}}{2}$ (E) $\frac{\sqrt{3}}{2}$

19. How many ordered pairs (x, y) of integers are solutions to the equation $\frac{xy}{x+y} = 2013$?

(A) 26 (B) 27 (C) 42 (D) 53 (E) 54

20. If $T\,E\,E\,N$ is a four-digit number such that $T\,E\,E\,N_5 + T\,E\,E\,N_7 = T\,E\,E\,N_8$, what is the value of the digit N?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

21. The vertices of a triangle are (12, 20), (26, 96) and (0, k). What value of k will give the smallest possible perimeter for this triangle?

(A) 40 (B) 42 (C) 44 (D) 46 (E) 48

22. Compute the ordered pair of positive integers (a, b) for which:

$$(2 - 1/2)(3 - 1/3)(4 - 1/4) \ldots (59 - 1/59)(60 - 1/60) = \frac{a!}{b!}$$

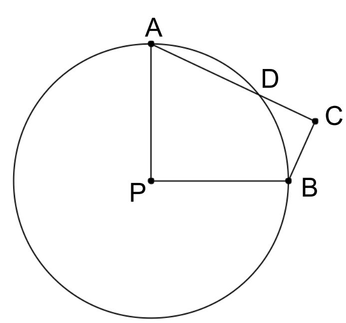
(A) (61, 5) (B) (61, 59) (C) (61, 60) (D) (60, 9) (E) (60, 59)

23. Compute the sum of all positive integral values of $x < 90$ which satisfy $\cos x = \sin(x^2)$, where x is measured in degrees.

(A) 19 (B) 64 (C) 107 (D) 119 (E) 128

24. The sum of the base 10 logarithms of the first five numbers of a geometric sequence is $8\frac{1}{3}$ and the sum of the base 10 logarithms of the next four numbers is $3\frac{2}{3}$. If the fourth number in the geometric sequence is $\sqrt{K}$, compute K.

(A) 120 (B) 150 (C) 600 (D) 1,000 (E) 1,200



25. In the diagram at the right radii $\overline{PA}$ and $\overline{PB}$ of circle P are perpendicular. Point D is chosen on minor arc AB so that AD = 6. Chord $\overline{AD}$ is extended beyond D to a point C such that $\overline{BC}$ is perpendicular to $\overline{AC}$. If DC = 5, compute the distance from P to C.

(A) $4\sqrt{2}$ (B) $5\sqrt{2}$ (C) $6\sqrt{2}$ (D) $7\sqrt{2}$ (E) $8\sqrt{2}$

THE 2013-2014 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

SOLUTIONS

1. A Debbie gets twice as many marbles as Don, so the total number of marbles the two possess must be a multiple of 3. Then the difference between 163 (the total number of marbles) and the number of chipped marbles must be divisible by 3. Of the six given numbers, only 22 has this property ($163 - 22 = 141 = 3\cdot 47$). Therefore, there are **22** chipped marbles. (Debbie got the bags with 27, 32, and 35 marbles, Don got 21 and 26).

2. E Notice that all multiples of 8 are in column 2. Since (8)(251) = 2008, the row containing 2008 looks very much like the row containing 8. Thus, 2013 is in column 5.

Column	1	2	3	4	5
	2009	2008	2007	2006	
		2010	2011	2012	2013

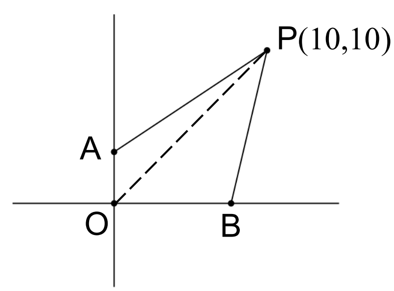
3. B Let m = the number of men and w = the number of women.

Then $35m + 25w = 31(m + w)$. Simplifying, $4m = 6w$ so that $\frac{m}{w} = \frac{3}{2}$.

4. A We are given $\frac{X^3 + Y^3}{X^3 + Z^3} = \frac{X + Y}{X + Z}$. Factoring the numerator and denominator of the left we get $\frac{(X+Y)(X^2 - XY + Y^2)}{(X+Z)(X^2 - XZ + Z^2)} = \frac{X+Y}{X+Z} \Rightarrow X^2 - XY + Y^2 = X^2 - XZ + Z^2$, from which $Y^2 - Z^2 = XY - XZ \Rightarrow (Y - Z)(Y + Z) = (Y - Z)X$. Since Y and Z are distinct, $X = Y + Z$.

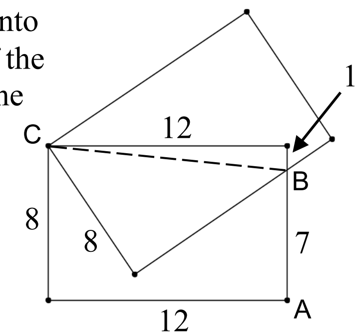
5. B Since the altitudes of triangles AOP and BOP from point P are equal (both 10) the ratio of their areas is $\frac{AO}{OB} = \frac{1}{2}$. Therefore, the ratio of the area of triangle AOP to the area of the quadrilateral is 1:3.

Using A(0, y), ½(10y) = $\frac{1}{3}$(30) or y = 2.



6. A If there are T states larger than Florida, there will be $49 - T$ smaller states. Similarly, if there are $2T$ states larger than Hawaii, there will be $49 - 2T$ smaller states. Therefore, $4(49 - 2T) = 49 - T$ and $T = 21$. Hence, Florida is the 22nd largest state and Hawaii is the 43rd largest. Therefore, 20 states are larger than Hawaii and smaller than Florida.

7. D Constructing line segment $\overline{CB}$ divides the overlapping region into two right triangles whose areas can be added to find the area of the region. The area of the upper triangle is ½(12)(1) = 6. Using the Pythagorean Theorem on the upper triangle, CB = $\sqrt{145}$. Using the Pythagorean Theorem on the lower triangle, the missing leg is 9, making the area of the lower triangle ½(8)(9) = 36. Therefore the desired area is 6 + 36 = 42.



8. D $100A + 10B + C - (A + B + C) = 99A + 9B = 9(11A + B)$. This will be a perfect square only if $11A + B$ is a perfect square. The chart shows the possible combinations. The only ones for which $11A + B$ is not a perfect square are 6 and 8. Since 8 is not among the choices, the correct answer is 6.

A	B	$11A+B$
1	5	16
2	3	25
3	3	36
4	5	49
5	9	64
6		None
7	4	81
8		None
9	1	100

9. B The discriminant of $ax^2 + bx + c = 0$ is $D = b^2 - 4ac$. Since all of the choices are odd numbers, and $4ac$ is even, then b^2, and hence b, must be odd. Let $b = 2n + 1$ where n is an integer. Then

$$b^2 - 4ac = (2n+1)^2 - 4ac = 4n^2 + 4n - 4ac + 1$$

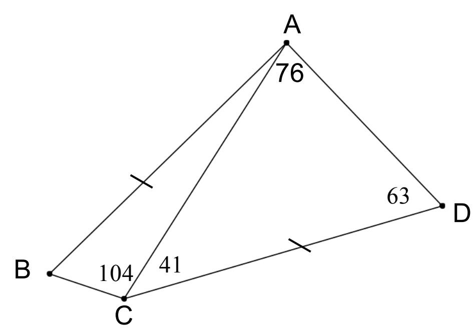
Therefore, the discriminant D must be one more than a multiple of 4. Of the choices, only 99 does not meet this requirement.

10. D First note that $2^0 + 2^3 + 2^4 = 1 + 8 + 16 = 25$ (a perfect square). No smaller sum of powers of 2 is a perfect square. Therefore, if we let $a = 102$, $b = 105$, and $c = 106$, then

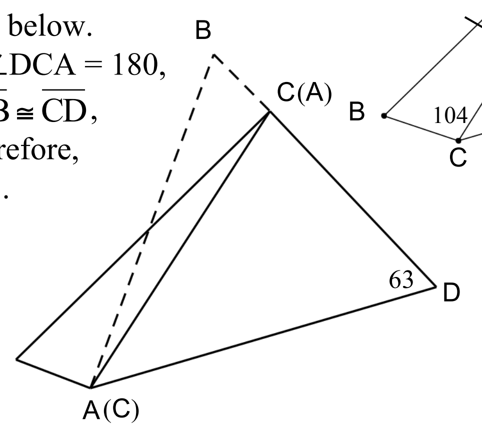
$N = 2^a + 2^b + 2^c = 2^{102} + 2^{105} + 2^{106} = 2^{102}(2^0 + 2^3 + 2^4) = 2^{102}(25)$ which is a perfect square. Therefore $a + b + c = 102 + 105 + 106 = 313$.

(Note 1: $2^{102} + 2^{104} + 2^{104} = 2^{102}(2^0 + 2^2 + 2^2) = 2^{102}(9)$ is a perfect square. Thus choice C would work if a, b, and c did not need to be distinct.)

(Note 2: $2^{102} + 2^{106} + 2^{108} = 2^{102}(2^0 + 2^4 + 2^6) = 2^{102}(81)$ is a perfect square. Thus choice E would work if the question did not ask for the smallest.)



11. B m∠DAC = 76. Note that ∠DAC is supplementary to ∠BCA. "Flip" triangle ABC so that points C and A are reversed. The diagram then looks like the one below. In this new diagram, m∠BCA + m∠DCA = 180, so that BAD is a triangle. Since $\overline{AB} \cong \overline{CD}$, ΔBAD is an isosceles triangle. Therefore, m∠ABC= 63, so that m∠BAC = 13.

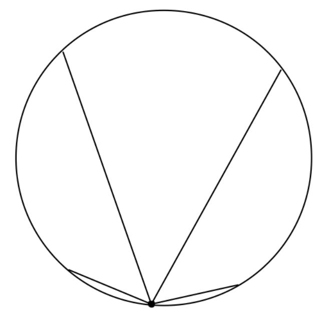


12. C Positive integral powers of 3 end in 3, 9, 7, and 1 only, with equal frequency as the exponent goes from 1 to 100. Consider the table of the sums of the units digits shown. To be a multiple of 5, a zero units digit must be 0 or 5. This occurs 4 times out of 16, for a probability of ¼.

+	3	9	7	1
3	6	2	0	4
9	2	8	6	0
7	0	6	4	8
1	4	0	8	2

13. C Suppose before scoring 98 and 70, the student's average grade on her first N tests was M. Then $\frac{NM+98}{N+1}=M+1$ and $\frac{NM+98+70}{N+2}=M+1-2$.
The first equation simplifies to $M+N=97$ and the second to $2M-N=170$. Thus $M=89$ and $N=8$. Including the two tests, she took **10** tests.

14. C Any regular polygon can be inscribed in a circle. Consider the the circle at the right in which two adjacent sides of a regular polygon and the angle trisectors of the angle between them have been drawn. Since the three inscribed angles intercept congruent arcs, each of the three arcs must contain the same number of sides of the regular polygon. Let this number of sides be k. Therefore, the number of sides of the regular polygon must be of the form $3k+2$. Thus the correct choice must be 2 more than a multiple of 3. Only 80 has this property.



15. A We are given $i+2i^2+3i^3+4i^4+\ldots+2013i^{2013}=A+Bi$.
Group the terms on left side of the equation in groups of four, and simplify in each group:

$(i-2-3i+4)+(5i-6-7i+8)+\ldots+(2009i-2010-2011i+2012)+2013i^{2013}$

The sum of each group of four terms is $2-2i$, and there are 503 groups. Also, $2013i^{2013}=2013i$. Therefore, the required sum is

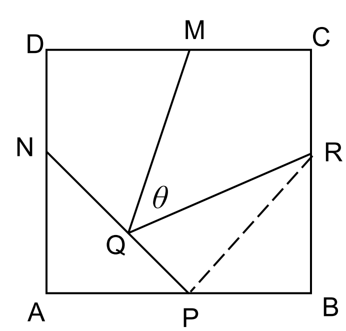
$$503(2-2i)+2013i=1006-1006i+2013i=1006+1007i.$$

So that $(A, B) = (1006, 1007)$.

16. E Construct segment $\overline{\text{RP}}$. Noting that ∠RPQ is a right angle, and that ∠RQP ≅ ∠MQN and RP = NP = 2(PQ), $\tan\angle\text{RQP}=\frac{\text{RP}}{\text{PQ}}=\frac{2}{1}$.

Noting that $\theta = 180 - 2(m\angle\text{RQP})$, and

$\tan 2(\angle\text{RQP})=\frac{2(\tan\angle\text{RQP})}{1-\tan^2(\angle\text{RQP})}=\frac{2\cdot 2}{1-2^2}=-\frac{4}{3}$.



Therefore, $\tan\theta=\tan[180-2(\angle\text{RQP})]=$

$$\frac{\tan 180-\tan 2\angle\text{RQP}}{1+(\tan 180)(\tan 2\angle\text{RQP})}=\frac{0-\left(-\frac{4}{3}\right)}{1}=\mathbf{\frac{4}{3}}$$

17. B Rewrite the formula as $f(n) - f(n-1) = n$. Look at the following pattern:

$$f(2013) - f(2012) = 2013$$
$$f(2012) - f(2011) = 2012$$
$$f(2011) - f(2010) = 2011$$
$$f(2010) - f(2009) = 2010$$
$$f(2009) - f(2008) = 2009$$
$$\vdots$$
$$f(2) - f(1) = 2$$

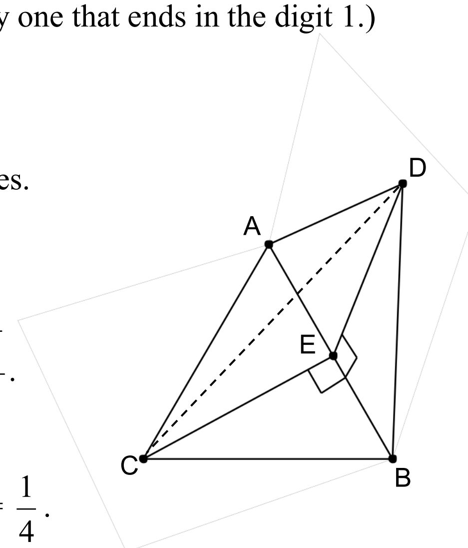
Adding all these equations,
$f(2013) - f(1) = 2013 + 2012 + 2011 + \ldots + 2$

Therefore, $f(2013) = 2013 + 2012 + 2011 + \ldots + 2 + 1$

$$= \frac{(2014)(2013)}{2} = (1007)(2013) = 2027091$$

(Note: calculation can be held to a minimum here since the choice B is the only one that ends in the digit 1.)

18. B Let a represent the length of the sides of the equilateral triangles.



Then the lengths of altitudes CE and DE are $\frac{a\sqrt{3}}{2}$.

Since CE and DE are in perpendicular planes, triangle CED is an isosceles right triangle. Therefore, CD = $\frac{a\sqrt{3}}{2}\sqrt{2} = \frac{a\sqrt{6}}{2}$.

Using the Law of Cosines on isosceles triangle ACD,

$$\left(\frac{a\sqrt{6}}{2}\right)^2 = a^2 + a^2 - 2a^2 \cos\angle CAD$$ from which $\cos\angle CAD = \frac{1}{4}$.

19. D The given equation $\frac{xy}{x+y} = 2013$ can be rewritten as $xy - 2013x - 2013y = 0$.
Adding 2013^2 to both sides allows us to factor the left side: $(x - 2013)(y - 2013) = 2013^2$.
Letting $(x - 2013) = k$, then $y - 2013 = \frac{2013^2}{k}$. Each solution (x, y) to the original equation yields a different value of k. Since $2013 = (3^1)(11^1)(61^1)$, we know that 2013^2 has $(2 + 1)(2+ 1)(2 + 1) = 27$ positive integral factors. Therefore, there are $2(27) = 54$ integral values of k. However, if $k = -2013$, $x = y = 0$ which does not work in the original equation. Therefore, there are 53 solutions.

20. C Noting that all three digits must be either 0, 1, 2, 3, or 4,

$TEEN_5 = 125T + 25E + 5E + N = 125T + 30E + N$

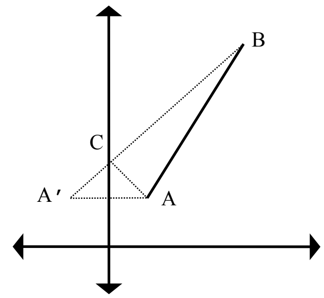
$TEEN_7 = 343T + 49E + 7E + N = 343T + 56E + N$

$TEEN_8 = 512T + 64E + 8E + N = 512T + 72E + N$

Therefore, $125T + 30E + N + 343T + 56E + N = 512T + 72E + N$, from which

$44T - 14E = N$. If $T = 1$, then $E = 3$ and $N = 2$. If $T = 2$, 3 or 4, then N must be greater than 4 for E to be less than 5. Therefore, N = 2.

21. C Let point A have coordinates (12, 20) and B have coordinates (26, 96). The length of AB is fixed for any choice of k on the y-axis. Reflect A over the y-axis to A′(-12, 20) and draw A′B intersecting the y-axis at C. Since AC = A′C, the distance A′B = BC + AC. Since the shortest distance between two points is along a straight line, BC + AC is a minimum. To find k, find the equation of A′B (y = 2x + 44) and k = 44.



22. A The left expression can be rewritten as $(\frac{3}{2})(\frac{8}{3})(\frac{15}{4})(\frac{24}{5})(\frac{35}{6})\ldots$. Dividing each numerator by the denominator of the *preceding* fraction, this expression becomes $(\frac{3}{1})(\frac{4}{1})(\frac{5}{1})(\frac{6}{1})(\frac{7}{1})\ldots(\frac{61}{60})$. If the numerator was multiplied by 2, it would become 61!. Fortunately, if the denominator is multiplied by 2, it becomes 120 = 5!. Therefore, the original expression is equivalent to $\frac{61!}{5!}$. Hence $(a, b) = (61, 5)$.

23. E To find solutions, note that for $0 < A < 90$, $\cos(A) = \sin(90 - A)$. Therefore, solutions may be obtained from $x^2 + x = 90$. This equation has one positive solution, $x = 9$ (the other solution is -10). Also, since $\sin[180 - (90 - A)] = \sin(90 - A) = \cos(A)$, solutions may be obtained from $180 - x^2 + x = 90$ or $x^2 - x - 90 = 0$. This also yields one positive solution, $x = 10$ (the other solution is -9). Hence two solutions are $x = 9, 10$.
To find any other values we need to find all possible integer solutions between 0 and 90 for the equations $x^2 \pm x = 90 + 360K$ (K a positive integer). Since $90 + 360K$ is divisible by 9 and 5 and $x^2 \pm x$ factors as $x(x \pm 1)$, we need to find factor pairs such that x is a multiple of 9 and x, $x + 1$, or $x - 1$ is a multiple of 5. The possibilities (other than (9,10)) are (35,36), (44,45), (45,46) (54,55), and (80,81). Of these pairs, only 54 and 55 have a product of the form $90 + 360K$. Therefore, the only other two solutions are $x = 54, 55$. Thus the sum of all four solutions is 9 + 10 + 54 + 55 = **128**.

24. D Let the first nine terms of the geometric sequence be $a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, ar^7, ar^8$
Adding the logarithms of the first five terms gives

$$\log a + \log ar + \log ar^2 + \log ar^3 + \log ar^4 = 5\log a + 10\log r = 5(\log a + 2\log r) = 5\log ar^2$$

Therefore, $5\log ar^2 = 8\frac{1}{3} = \frac{25}{3} \Rightarrow \log ar^2 = \frac{5}{3}$.

Thus the log of the 3rd term of the sequence is $\frac{5}{3}$.

Similarly, the sum of the logs of the all nine terms gives

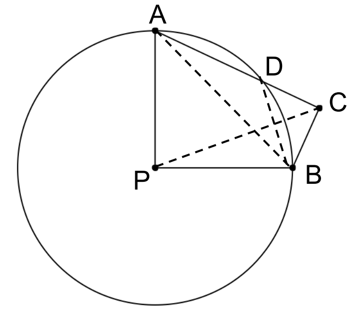
$$9\log a + 36\log r = 9\log ar^4 = 8\frac{1}{3} + 3\frac{2}{3} = 12 \Rightarrow \log ar^4 = \frac{4}{3}.$$

Thus the log of the 5th term of the sequence is $\frac{4}{3}$.

Since the logs of the numbers in a geometric sequence form an arithmetic sequence, the log of the 4th term of the sequence is the average of the logs of the 3rd and 5th terms.

Therefore, $\log ar^3 = \frac{1}{2}\left(\frac{4}{3} + \frac{5}{3}\right) = \frac{3}{2}$ making the 4th term is $10^{\frac{3}{2}} = \sqrt{1000}$ and $K = 1000$.

25. E Method 1



Construct chord $\overline{BD}$. Since the measure of minor arc ADB = 90, the measure of major arc AB = 270, making the measure of inscribed angle ADB = 135. Therefore, m∠CDB = 45, so that Δ DCB is an isosceles right triangle, and DC = CB = 5.

Since both pairs of opposite angles of quadrilateral APBC are supplementary, it is a cyclic quadrilateral.

Although it is possible to compute the radius of circle P ($\sqrt{73}$), it is not necessary. Letting AP = PB = R, and noting that AB = R$\sqrt{2}$, apply Ptolemy's Theorem.
5R + 11R = PC(R$\sqrt{2}$). Therefore, PC = $\frac{16}{\sqrt{2}} = 8\sqrt{2}$.

Method 2

Construct chord $\overline{BD}$. Since the measure of minor arc ADB = 90, the measure of major arc AB = 270, making the measure of inscribed angle ADB = 135. Therefore, m∠CDB = 45, so that Δ DCB is an isosceles right triangle, and DC = CB = 5.

Using the Pythagorean Theorem on ΔABC, AB = $\sqrt{146}$. Since ΔAPB is an isosceles right triangle, AP = PB = $\sqrt{73}$.

Since ∠PAC and ∠PBC are supplementary let m∠PAC = θ, and m∠PBC = 180 – θ.
Using the Law of Cosines on both ΔAPC and ΔBPC:

(1) $PC^2 = 73 + 121 - (2)(11)\sqrt{73}\cos\theta = 194 - 22\sqrt{73}\cos\theta$

(2) $PC^2 = 73 + 25 - (2)(5)\sqrt{73}\cos(180-\theta) = 98 - 10\sqrt{73}\cos(180-\theta) = 98 + 10\sqrt{73}\cos\theta$

Subtracting (2) from (1) and solving for $\cos\theta$ we get $\cos\theta = \frac{3}{\sqrt{73}}$.

Substituting this value into equation (2) we get $PC^2 = 128$ from which PC = $8\sqrt{2}$.