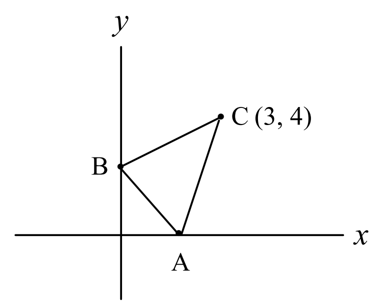
THE 2013-2014 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

In addition to scoring student responses based on whether a solution is correct and complete, consideration will be given to elegance, simplicity, originality, and clarity of presentation.

Calculators are NOT permitted.

1. *A* and *B* both represent nonzero digits (not necessarily distinct). If the base ten numeral $\underline{A}\,\underline{B}$ divides, without remainder, the base ten numeral $\underline{A}\,\underline{0}\,\underline{B}$ (whose middle digit is zero), find, with proof, all possible values of $\underline{A}\,\underline{B}$.

2. A and B are points on the positive *x* and positive *y* axes respectively and C is the point with coordinates (3, 4). Prove that the perimeter of triangle ABC is greater than 10.



3. One solution for the equation $a^2 + b^2 + c^2 + 2 = abc$ is $a = 3$, $b = 3$ and $c = 4$.

   a. Find a solution (a, b, c) where a, b, and c are integers all larger than 10.

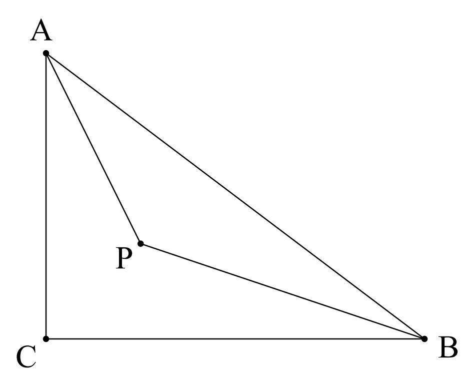
   b. Prove that there are infinitely many solutions (a, b, c) where a, b, and c are positive integers.

4. Consider the equation $\sqrt{x} = \sqrt{a} + \sqrt{b}$, where x is a positive integer.

   a. Prove that the equation has a solution (a, b) where a and b are both positive integers, if and only if x has a factor which is a perfect square greater than 1.

   b. If $x \le 1{,}000$, compute, with proof, the number of values of x for which the equation has at least one solution (a, b) where a and b are both positive integers.

5. In right triangle ABC, AC = 6, BC = 8 and AB = 10. PA and PB bisect angles A and B respectively. Compute, with proof, the ratio $\frac{\text{PA}}{\text{PB}}$.



SOLUTIONS – KSU MATHEMATICS COMPETITION – PART II 2013–14

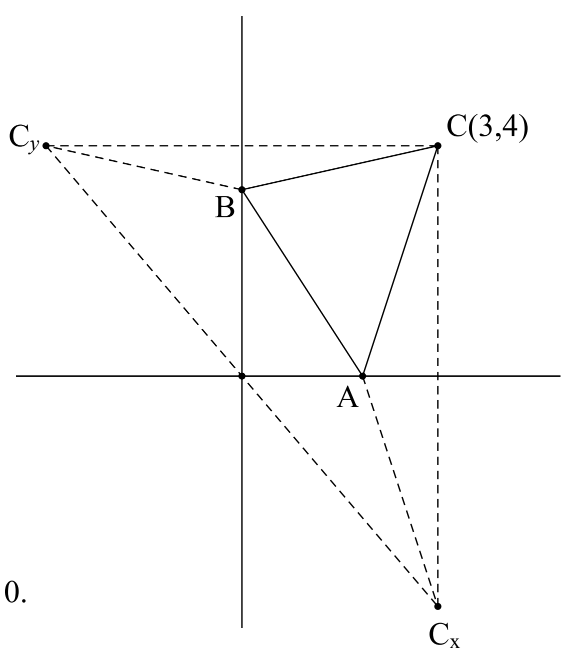
1. Of course, this problem can be done by trial and error (there are only 81 possibilities), but we present a more elegant solution.

   Suppose $\dfrac{A0B}{AB} = k$. Then $100A + B = 10Ak + Bk$ or
   (i) $100A - 10Ak = Bk - B = B(k-1)$

   Since the left side of equation (i) is a multiple of 5, the right side must also be. Since $1 < k < 10$, the right side is positive and thus so is the left side. Then either 5 divides $k - 1$ or 5 divides B.

   Suppose 5 divides $k - 1$. Then $k = 6$, so that (i) becomes $40A = 5B$, or $B = 8A$. Therefore, $A = 1$, $B = 8$, and $\underline{A}\,\underline{B} = 18$.

   Now suppose 5 divides B. Then $B = 5$, and (i) becomes $10A(10 - k) = 5(k - 1)$, or $2A(10 - k) = k - 1$. From this, $A = \dfrac{k-1}{2(10-k)}$. Since the denominator is even, $k - 1$ must be even and k is odd. Trying k = 3, 5, 7, and 9, we find only $\underline{A}\,\underline{B}$ = 15 and 45 corresponding to k = 7, 9 respectively. Therefore, the only possible values for $\underline{A}\,\underline{B}$ are 15, 18, and 45.



2. Consider the reflection images of C over the x and y axes. Call these points C_x and C_y, respectively, as shown. The coordinates of C_x are (3, -4) and of C_y are (-3, 4). The length of $\overline{CC_x}$ is 2(4) = 8 and the length of $\overline{CC_y}$ is 2(3) = 6.

   Since ΔABC is a right triangle, the length of $\overline{C_xC_y}$ is $\sqrt{6^2+8^2}$ =10. Also note that because C_y is a reflection image of C, BC = BC_y. Similarly, AC = AC_x.
   In quadrilateral ABC_yC_x, $C_yB + BA + AC_x > \overline{C_xC_y} = 10$.
   Therefore, by substitution, BC + BA +AC > 10.

3. Suppose we begin with two positive integers a and b, and we try to find a third integer x such that $a^2 + b^2 + x^2 + 2 = abx$. Then the problem can be thought of as finding an integer solution (if one exists) for the quadratic equation $x^2 - (ab)x + (a^2 + b^2 + 2) = 0$.

   If there is some integer solution $x = c$, then there must exist a real number d such that

$$x^2 - (ab)x + (a^2 + b^2 + 2) = (x - c)(x - d) = x^2 - (c + d)x + cd$$

   Comparing the coefficients on the left and right sides of this last equation, we know that $ab = c + d$, so that $d = ab - c$ is also an integer. Therefore, given any three integers a, b, and c such that $a^2 + b^2 + c^2 + 2 = abc$, we can replace c with $ab - c$ to obtain another solution.

   We know that (4, 3, 3) is a solution. So we can replace one of the 3's with $3 \cdot 4 - 3 = 9$ to get the solution (4, 3, 9). Since a, b, and c are interchangeable, We can obtain other solutions by repeatedly replacing the smallest number (which we will call c) by $ab - c$. Hence, listing the numbers in decreasing order at each step, we obtain the following solutions:

   (4, 3, 3) ⟶ (9, 4, 3) ⟶ (33, 9, 4) ⟶ (293, 33, 9) ⟶ (9660, 293, 33).

   Since this process can be repeated indefinitely, there are infinitely many positive integer solutions (a, b, c) to the given equation.

4. (i) Given $\sqrt{x} = \sqrt{a} + \sqrt{b}$.

Suppose $x = k^2 y$, with k and y positive integers, and $k > 1$. We must prove that there exists at least one pair of positive integers (a, b) that satisfies the equation.

We have $\sqrt{x} = \sqrt{k^2 y} = k\sqrt{y}$. Since $k > 1$, then $k - 1 > 0$. Therefore,

$$\sqrt{x} = k\sqrt{y} = (k-1)\sqrt{y} + \sqrt{y} = \sqrt{(k-1)^2 y} + \sqrt{y}.$$

Since both $(k-1)^2 y$ and y are both positive integers, setting $a = (k-1)^2 y$ and b = y gives the desired result.

Now suppose a and b are both positive integers that satisfy $\sqrt{x} = \sqrt{a} + \sqrt{b}$.
We must show that x has a perfect square factor greater than 1.

$$\sqrt{x} = \sqrt{a} + \sqrt{b} \;\Rightarrow\; x = \left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$$

Since x is a positive integer, $\sqrt{ab}$ must be a perfect square. There are two possibilities: either (1) a and b are both perfect squares or (2) the non-square factors of a and b are equal.

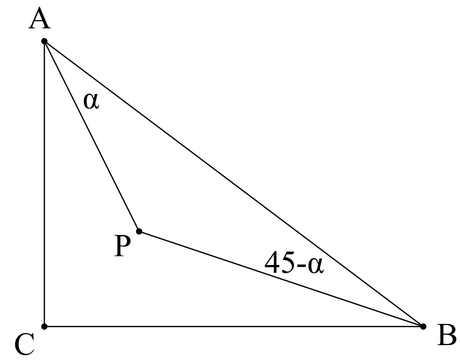
1) If a and b both perfect squares, let $a = m^2$ and $b = n^2$. Then
$$x = a + b + 2\sqrt{ab} = m^2 + n^2 + 2mn = (m+n)^2.$$
Therefore, x has a perfect square factor.

2) If the non-square factors of a and b are equal, let $a = m^2 p$ and b = $n^2 p$. Then
$$x = a + b + 2\sqrt{ab} = m^2 p + n^2 p + 2mnp = p(m+n)^2$$
and again, x has a perfect square factor.

Therefore, the equation has a solution (a, b) where a and b are both positive integers, if and only if x has a factor which is a perfect square greater than 1.

(ii) There are 250 values of $x \le 1000$ that contain a factor of 4. Similarly, the number of values of x ≤ 1000 that, respectively, contain a factor of $3^2, 5^2, 7^2, 9^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2$ is 111, 40, 20, 8, 5, 3, 2, 1, 1, and 1, for a total of 442. However, some values, like 36 = $(2^2)(3^2)$, have been counted twice and must be subtracted from our total. The number of values of x ≤ 1000 that, respectively, contain a factor of $(2^2)(3^2)$, $(2^2)(5^2)$, $(2^2)(7^2)$, $(2^2)(11^2)$, $(2^2)(13^2)$, $(3^2)(5^2)$, and $(3^2)(7^2)$ is 27, 10, 5, 2, 1, 4, and 2, a total of 51 such duplicates. However, the factor $(2^2)(3^2)(5^2)$ was counted three times, once in each group. Therefore, the final total is 442 – 51 + 1 = 392.

5. Method 1:



We will refer to ∠CAB as ∠A and ∠CBA as ∠B.
So that m∠A + m∠B = 90°.

Then m∠P = 180 – ½ (m∠A + m∠B) = 135°.
So that, m∠PAB + m∠PBA = 45. Represent the measures of these two angles with α and 45 – α.

Using the Law of Sines on ΔAPB

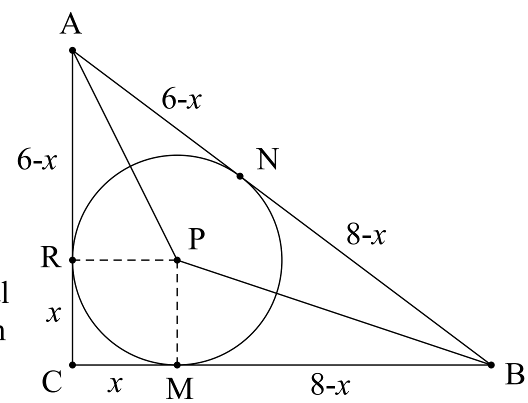
$$\frac{PA}{PB} = \frac{\sin(45-\alpha)}{\sin\alpha} = \frac{\sin 45\cos\alpha - \cos 45\sin\alpha}{\sin\alpha} = \sin 45\cot\alpha - \cos 45.$$

Now $\cot\alpha = \cot(\tfrac{1}{2}A) = \dfrac{1+\cos A}{\sin A}$ (using the appropriate half-angle formula)

But in ΔABC, $\cos A = \dfrac{6}{10}$ and $\sin A = \dfrac{8}{10}$, making $\cot\alpha = \dfrac{1+\frac{6}{10}}{\frac{8}{10}} = 2.$

Finally, $\dfrac{PA}{PB} = (\sin 45)(2) - \cos 45 = \dfrac{\sqrt{2}}{2}(2) - \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2}}{2}.$

Method 2:



Note that since point P is the intersection of the angle bisectors of ΔABC, P is the incenter (the center of the inscribed circle).

Noting that the tangent segments to a circle from an external point are congruent, represent the lengths of the segments in the diagram as shown.

Then $6 - x + 8 - x = 10$ and $x = 2$.
Therefore, right ΔARP has side lengths 2, 4, and $2\sqrt{5}$, and right ΔBMP has side lengths 2, 6, and $2\sqrt{10}$.

Therefore, $\dfrac{PA}{PB} = \dfrac{2\sqrt{5}}{2\sqrt{10}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.