THE 2014–2015 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. A *prim-prime* is a prime number that can be expressed as the sum of two other prime numbers. What is the sum of the biggest *prim-prime* less than 100 and the smallest *prim-prime* that exists?

   (A) 98 (B) 94 (C) 78 (D) 66 (E) None of these

2. The eighth term of an arithmetic sequence is 48. If the sum of the ninth and tenth terms is 75, compute the value of the eleventh term of the arithmetic sequence.

   (A) 27 (B) 29 (C) 31 (D) 33 (E) 35

3. In the rectangular coordinate plane, any circle which passes through the points with coordinates (20, 13) and (5, y) cannot also pass through the point with coordinates (11, 16). Compute y.

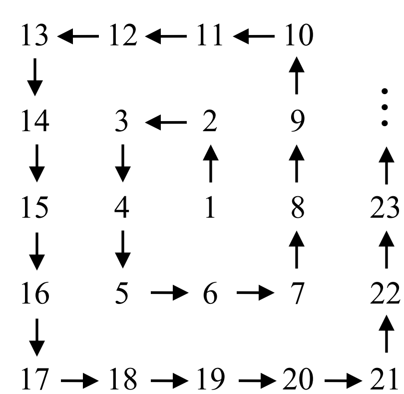
   (A) 12 (B) 14 (C) 16 (D) 18 (E) None of these

4. A quadratic polynomial f satisfies $f(x) \geq 0$ for all x. If $f(1) = 0$ and $f(3) = 3$, compute $f(5)$.

   (A) 5 (B) 7 (C) 8 (D) 10 (E) 12

5. If the positive integers are arranged in the pattern shown in the diagram, what number would appear to the immediate right of 2014?

   (A) 2180 (B) 2186 (C) 2191 (D) 2197 (E) 2208



6. Let the elements of $S = \{10, 11, 12, \ldots, 100\}$ be the perimeters of triangles with integral side lengths. For how many of these perimeters are there no triangles with a side of length 1?

(A) 44 (B) 45 (C) 46 (D) 47 (E) 48

7. Suppose that B is a subset of A. There are precisely 120 subsets of A which are not subsets of B. How many members does set A have?

(A) 5 (B) 7 (C) 8 (D) 10 (E) 12

8. Consider the equation $(3 + 4k)x + (2 - k)y = 3$. For each value of k there are many pairs of values of x and y which satisfy the equation. However, there is only one pair of values (x, y) which satisfies this equation no matter what value is assigned to k. Compute $x + y$ for this pair.

(A) $\frac{5}{4}$ (B) $\frac{7}{5}$ (C) $\frac{11}{7}$ (D) $\frac{15}{11}$ (E) $\frac{23}{15}$

9. The year 1868 was *peculiar* in that the number formed by the first two digits added to the number formed by the last two digits gives the number formed by the middle two digits (i.e. $18 + 68 = 86$). The year 1978 was also *peculiar* ($19 + 78 = 97$). The year 2014 is not *peculiar* because $20 + 14 \neq 01$ (or just 1). Compute the sum of the next two *peculiar* years after 2014.

(A) 4683 (B) 4725 (C) 4879 (D) 4917 (E) None of these

10. A rectangular sheet of paper measuring 8 ½″ by 11″ is folded twice. In the first fold (Figure I), $\overline{AB}$, formerly an edge, now lies along edge $\overline{BC}$. The second fold (Figure II) is along the dotted line. Compute the number of square inches in the area of the shaded rectangle in Figure II.

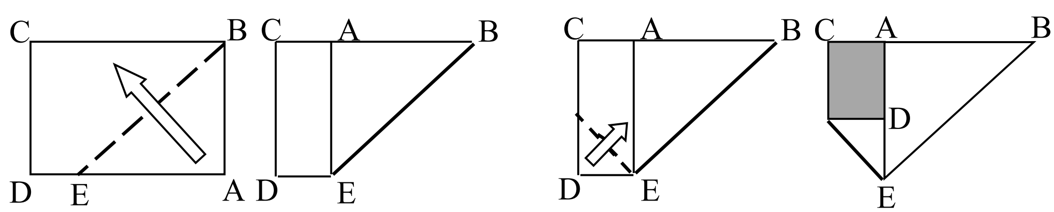


Figure I Figure II

(A) 23.3 (B) 18.4 (C) 15.0 (D) 12.5 (E) 8.6

11. The Jones family has five children, and the Smith family has three children. Of the eight children there are five girls and three boys. Let $\frac{m}{n}$ be the probability that one of the families has only girls for children. Given that m and n are relatively prime positive integers, compute $m + n$.

(A) 57 (B) 67 (C) 347 (D) 396 (E) 397

12. In racing over a given distance d yards, Abe beat Barbara by 20 yards, Barbara beat Chris by 10 yards, and Abe beat Chris by 28 yards. If each person ran at a constant speed, compute d.

(A) 100 (B) 120 (C) 150 (D) 200 (C) 240

13. There are two values of m for which the difference between the roots of the equation $mx^2 + 5x - 6 = 0$ is 1. Compute the sum of these two values of m.

(A) 12 (B) 15 (C) 18 (D) 20 (E) 24

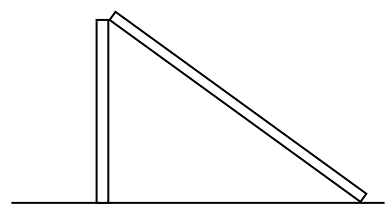
14. How many values of x, $0° < x < 360°$, satisfy the equation $\sin 3x = \cos 7x$?

(A) 4 (B) 8 (C) 10 (D) 12 (E) 14

15. Let N = 374_b where b is the smallest base for which N is a perfect square. Then the value of N in base 10 is

(A) 400 (B) 576 (C) 784 (D) 1024 (E) 1156

16. Lightning struck and broke a telephone pole in front of my house the other night (see diagram). I noticed that the top touched the ground at a distance from the base that was half the pole's original height. How far up the pole did the break occur?



(A) $\frac{1}{2}$ of the way up (B) $\frac{2}{5}$ of the way up (C) $\frac{3}{8}$ of the way up

(D) $\frac{5}{16}$ of the way up (E) $\frac{11}{32}$ of the way up

17. Don and Debbie each have forty coins consisting of nickels, dimes, and quarters. Each has exactly \$5.00. If Don has twice as many quarters as Debbie, how many nickels does Debbie have?

(A) 1 (B) 4 (C) 7 (D) 10 (E) 13

18. How many pairs of positive integers (a, b) are there such that a < b and $\frac{1}{a} + \frac{1}{b} = \frac{1}{2014}$?

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

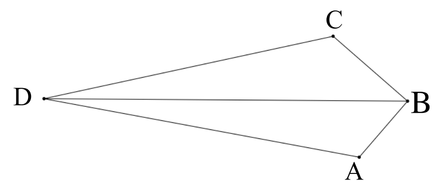
19. Let $z = a + bi$ where a and b are positive integers and i is the imaginary unit. Compute the smallest possible value of $a + b$ for which $z + z^2 + z^3$ is a real number.

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

20. There is one value of k for which the system of equations $r^2 + s^2 = t$ and $r + s + t = k$ will have exactly one real solution (r, s, t). What is this value of k?

(A) -3 (B) $-\frac{3}{2}$ (C) -1 (D) $-\frac{1}{2}$ (E) None of these

21. In the diagram, AB = 3, BC = 4, CD = 12, AD = 13, and $\overline{AB} \perp \overline{BC}$. Compute the ratio of the area of ΔABD to the area of ΔBCD.



(A) $\frac{3}{4}$ (B) $\frac{4}{5}$ (C) $\frac{5}{6}$ (D) $\frac{6}{7}$ (E) $\frac{7}{8}$

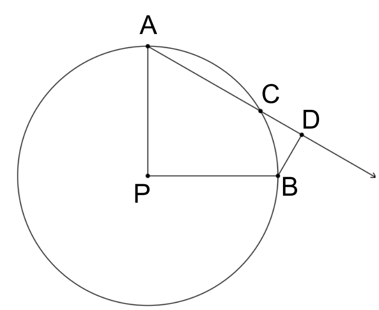
22. Dr. Garner presented a challenge to his math class. He wrote a set of consecutive positive integers, beginning with 1, on a piece of paper. Then he erased one of the numbers and told his students that the average of the remaining numbers was $35\frac{7}{17}$. Dr. Garner challenged his students to identify the number he erased. What number did he erase?

(A) 7 (B) 11 (C) 17 (D) 23 (E) 29

23. How many ordered pairs (x, y) of positive integers are there such that both x and y are less than 100, and such that $\log_{10} x + \log_{10} y$ is an integer?

(A) 8 (B) 10 (C) 12 (D) 14 (E) 16

24. In circle P, radii $\overline{PA}$ and $\overline{PB}$ are perpendicular. Point C is chosen on minor arc AB so that the length of chord $\overline{AC}$ is equal to the radius of the circle. $\overline{AC}$ is extended through point C and $\overline{BD}$ is constructed perpendicular to ray AC. If the radius of the circle is 4, compute the length of $\overline{CD}$.



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $3\sqrt{2}-3$ (D) $2\sqrt{3}-2$ (E) $3\sqrt{2}-2\sqrt{3}$

25. A seven-digit number N may be formed by writing, in any order, the digits 1, 2, 3, 4, 5, 6, 7 once each (e.g. 1,235,476). How many numbers N of this type are exactly divisible by 11?

(A) 484 (B) 576 (C) 676 (D) 784 (E) None of these

Solutions

1. C Clearly, the smallest *prim-prime* is 2 + 3 = 5. The largest prim-prime must be odd, and so it must be a prime number plus 2. Looking at the primes less than 100, the largest is 73 = 71 + 2. Therefore, the desired answer is 5 + 73 = 78.

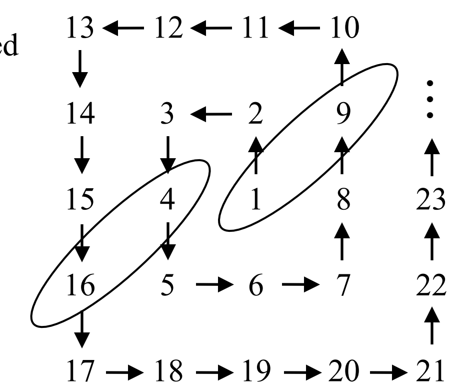
2. A The sum of the 8^{th}, 9^{th}, and 10^{th} terms of the sequence is 48 + 75 = 123. The average of these three terms is 41 and this is the 9^{th} term. Therefore, the three terms are 48, 41, and 34, making the 11^{th} term 34 – 7 = 27.

3. D A circle can be made to pass through any three non-collinear points. Therefore, the given points must be collinear. Using slopes, $\frac{13-y}{20-5} = \frac{13-16}{20-11}$. Solving for y, we obtain $y = 18$.

4. E Since $f(x) \geq 0$ for all x, and $f(1) = 0$, the quadratic polynomial must be of the form $f(x) = a(x-1)^2$. Substituting $x = 3$, we have $f(3) = 3 = 4a$, so that $a = \frac{3}{4}$. Finally,

$$f(5) = \frac{3}{4}(5-1)^2 = 12.$$

5. D Notice that the perfect squares lie along the two indicated paths. Since 2014 is 11 less than $45^2 = 2025$, it will lie along the sequence of numbers going up to the right, as shown below. Therefore, the number to the immediate right of 2014 is 2197.



1851	2026	47^2
1850	45^2	2208
43^2	2024	2207
⋮	⋮	⋮
	2014	2197

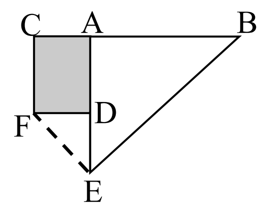
6. C For a triangle with an odd perimeter and a side of length 1, there is an isosceles triangle with two equal odd side lengths. For example, if a perimeter was 17, then the sides could be 1, 8, 8. If a perimeter was even and had one side of length 1, then the other two side lengths would sum to an odd number. But then those two side lengths would have to differ by at least 1. So, by the triangle inequality, no such triangle could be formed. Thus, no <u>even</u> perimeter can involve a triangle with a side of length 1. There are 46 even numbers in S = {10, 11, 12, . . . , 99, 100}.

7. B Let n be the number of members of set A and m the number of members of set B, $m < n$. Then A has 2^n subsets, and B has 2^m subsets. We are looking for two powers of 2 that differ by 120. The only possible choices are $2^3 = 8$ and $2^7 = 128$. Thus, $n = 7$ and A has 7 members.

8. D Expanding $(3 + 4k)x + (2 - k)y = 3$ we obtain $3x + 4kx + 2y - ky = 3$. Rearranging and regrouping terms yields $3x + 2y + k(4x - y) = 3$. This will be true for all values of k if the equations $4x - y = 0$ and $3x + 2y = 3$ are both true. Solving these two equations simultaneously gives $(x, y) = (\frac{3}{11}, \frac{12}{11})$. Therefore, $x + y = \frac{15}{11}$.

9. E Represent the four-digit number by ABCD. Then (10A + B) + (10C + D) = (10B + C). Rearranging terms, 10A + D = 9(B – C). Therefore, the first and last digits form a two-digit number that is a multiple of 9. If the next *peculiar* year is in this millennium, then A = 2, D = 7. Trial and error gives the next two *peculiar* years as 2307 (23 + 07 = 30) and 2417 (24 + 17 = 41). The required sum is 2307 + 2417 = 4724. Since this is not one of the choices, the answer is (E) None of these. (Interestingly, 1868 +1978 = 3846, which is also a *peculiar* year, but 2307 + 2417 = 4724 is one year before a *peculiar* year, 4725.)

10. C The area of the unfolded sheet of paper is (8.5)(11) = 93.5. The area of rectangle ACDE (the left part of Figure II) is $93.5 - 8.5^2 = 21.25$. Since DE = 2.5, the area of the small folded triangle FDE (at the right) is $\frac{1}{2}(2.5^2) = 3.125$. Therefore, the area of shaded rectangle ACFD is 21.25 – 2(3.125) = **15**.



11. B Since there are 8 children with three of them in the Smith family, there are

$${}_8C_3 = \frac{8!}{(5!)(3!)} = 56 \text{ equally likely collections of children for the Smith family.}$$

There are ${}_5C_3 = \frac{5!}{(3!)(2!)} = 10$ ways for the Smith family to have three girls.

There is only one way for the Smith family to have three boys so that the Jones children are all girls. Thus, there are 10 + 1 = 11 ways for one family to have only girls for children. This gives a probability of $\frac{11}{56}$ so that $m + n = 11 + 56 = 67$.

12. A Let A, B, and C be the respective speeds of Abe, Barbara, and Chris. Then

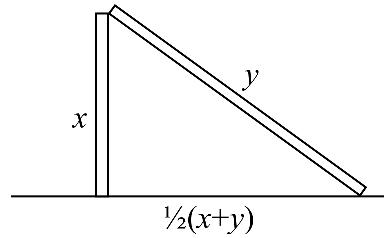
$$(1)\ \frac{d}{A} = \frac{d-20}{B} \qquad (2)\ \frac{d}{B} = \frac{d-10}{C} \qquad (3)\ \frac{d}{A} = \frac{d-28}{C}.$$

Therefore, from (1) and (3) $\frac{d-20}{B} = \frac{d-28}{C}$. Dividing this equation by $\frac{d}{B}$ and using (2), we obtain $\frac{d-20}{C} = \frac{d-28}{d-10}$, from which $d = 100$.

13. E Using the quadratic formula, the roots of the equation, in terms of m, are $\frac{-5 \pm \sqrt{25+24m}}{2m}$. Their difference is $\frac{\pm\sqrt{25+24m}}{m}$. Therefore, $\frac{\pm\sqrt{25+24m}}{m} = 1$, both of which lead to $25 + 24m = m^2$. This last equation has roots -1 and 25, which both satisfy the conditions of the problem. Therefore, the desired sum is 24.

14. E Since $\sin 3x = \cos(90 - 3x)$, $7x = 90 - 3x$ from which $10x = 90$, or more generally, $10x = 90 + 360k \Rightarrow x = 9 + 36k$. Since $0° < x < 360°$, there will solutions for $k = 0, 1, 2, 3, \ldots, 9$ for a total of 10 solutions.
Also, $\sin 3x = \cos(270 + 3x)$, $7x = 270 + 3x$ from which $4x = 270$, or more generally, $4x = 270 + 360k \Rightarrow x = 67.5 + 90k$. Thus there will be solutions for $k = 0, 1, 2, 3$, for an additional 4 solutions. Total number of solutions is 14.

15. C $374_b = 3b^2 + 7b + 4 = (3b+4)(b+1)$. Since $(3b+4) - 3(b+1) = 1$, $(3b+4)$ and $(b+1)$ have no common factors greater than 1 for all values of b. Therefore, the product will be a perfect square only if both 3b+4 and b+1 are perfect squares. Examining the first few integers for b > 7, we find that when b = 15, the factors become 49 and 16, both perfect squares, and $374_b = 28^2 = 784$ in base ten.



16. C Using the Pythagorean Theorem, $x^2 + \left[\frac{1}{2}(x+y)\right]^2 = y^2$,

from this, $x^2 + \frac{1}{4}(x^2 + 2xy + y^2) = y^2$. Combining like

terms, we obtain $5x^2 + 2xy - 3y^2 = 0$. Factoring,

$(5x - 3y)(x + y) = 0$ so that $5x = 3y$ or $x = \frac{3}{5}y$. Since the original pole's height was

$x + y = \frac{3}{5}y + y = \frac{8}{5}y$, the break occurred $\frac{3}{8}$ of the way up.

17. A For any one person, the information given can be translated into

$5n + 10(40-n-q) + 25q = 500$ which becomes $3q-n = 20$.

Make a chart of all the possibilities.

n	**1**	4	7	10	13	16	19	**22**	25
d	**32**	28	24	20	16	12	8	**4**	0
q	**7**	8	9	10	11	12	13	**14**	15

Since Don has twice as many quarters as Debbie, we need to look for two columns in which the number of quarters in one is twice that in another. The two columns are in bold print. Therefore, Debbie has **1** nickel.

18. E Certainly $a > 2014$. Let $a = 2014 + k$ for some positive integer k. Then

$$\frac{1}{b} = \frac{1}{2014} - \frac{1}{2014+k} = \frac{k}{2014(2014+k)} \Rightarrow b = 2014 + \frac{2014^2}{k}.$$

Since $b > a = 2014 + k$, then $\frac{2014^2}{k} > k$. Thus, $k < 2014$. Since b is an integer, k must be a factor of $2014^2 = (2^2)(19^2)(53^2)$. There are a total of 13 possibilities:

$1, 2, 19, 53, 2^2, 19^2, (2)(19), (2)(53), (19)(53), (2^2)(19), (2^2)(53), (2)(19^2), (2^2)(19^2)$.

19. E $z = a + bi$, $z^2 = (a+bi)^2 = a^2 - b^2 + 2abi$, and $z^3 = (a+bi)^3 = a^3 + 3a^2bi - 3ab^2 - b^3i$.

Therefore, $z + z^2 + z^3 = (a + a^2 + a^3 - b^2 - 3ab^2) + (b + 2ab + 3a^2b - b^3)i$.

For $z + z^2 + z^3$ to be a real number, $b + 2ab + 3a^2b - b^3 = b(1 + 2a + 3a^2 - b^2) = 0$.

Since b ≠ 0, $1 + 2a + 3a^2 = b^2$. Make a table of values.

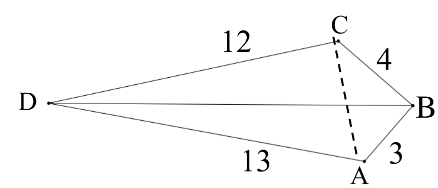
a	b^2	b
1	6	non-integer
2	17	non-integer
3	34	non-integer
4	57	non-integer
5	86	non-integer
6	121	11

Therefore, the smallest possible value for $a + b$ is 17 and $z = a + b\text{i} = 6 + 11i$.

20. D If (r, s, t) is a solution, then by symmetry (s, r, t) is also. The solution will be unique if and only if $r = s$. Hence, $2r^2 = t$ and $2r + t = k$. Therefore, $2r^2 + 2r - k = 0$. This is a quadratic equation with $a = 2$, $b = 2$, and $c = -k$. Setting the discriminant equal to zero (to ensure a unique value for r),

$b^2 - 4ac = 4 + 8k = 0$, and $k = -\frac{1}{2}$.

21. E Construct $\overline{AC}$ and note that ΔABC is a 3-4-5 right triangle, making ΔACD a 5-12-13 right triangle, with right angle ACD. If ∠ACB = θ, then



$$\text{area of } \Delta CBD = \frac{1}{2}(12)(4)\sin(90+\theta) = 24\cos\theta = 24\left(\frac{4}{5}\right) = \frac{96}{5}.$$

Since the area of quadrilateral ABCD is $\frac{1}{2}(3)(4) + \frac{1}{2}(5)(12) = 36$,

the area of ΔABD = $36 - \frac{96}{5} = \frac{84}{5}$ and the required ratio is $\frac{84}{96} = \frac{7}{8}$.

22. A Let n be the largest number Dr. Garner wrote on paper. The largest possible average is obtained when 1 is erased. The average is then

$$\frac{2+3+4+\ldots+n}{n-1} = \frac{\frac{(n+1)(n)}{2} - 1}{n-1} = \frac{n+2}{2}$$

The smallest average is obtained when n is erased. The average is then

$$\frac{1+2+3+\ldots+(n-1)}{n-1} = \frac{\frac{(n)(n-1)}{2}}{n-1} = \frac{n}{2}$$

Therefore, $\frac{n}{2} \le 35\frac{7}{17} \le \frac{n+2}{2} \quad \Rightarrow \quad 68\frac{14}{17} \le n \le 70\frac{14}{17}$. Thus, $n = 69$ or 70.

Since $35\frac{7}{17}$ is the average of $(n-1)$ integers, $35\frac{7}{17}(n-1)$ must be an integer.

Therefore, $n = 69$. If x is the number erased, then $\dfrac{\frac{1}{2}(69)(70) - x}{68} = 35\frac{7}{17}$ from which $x = 7$.

23. E Since $\log_{10} x + \log_{10} y = \log_{10} xy$ is an integer, $xy = 10^k$ where k is a nonnegative integer. Since x and y are less than 100, $k < 4$.

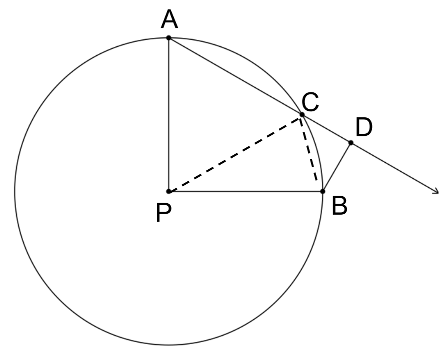
If $k = 0$, (x, y) = (1, 1)

If $k = 1$, $xy = 10$. Since 10 has four factors, there are four ordered pairs, namely (1, 10), (2, 5), (5, 2), and (10, 1).

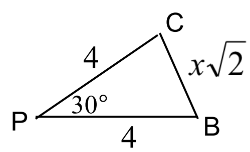
If $k = 2$, xy = 100 and there are seven ordered pairs, (2, 50), (4, 25), (5, 20), (10, 10), (20, 5), (25, 4), and (50, 2).

If $k = 3$, $xy = 1000$ and there are four ordered pairs, (20, 50), (25, 40), (40, 25), and (50, 20).

Therefore, altogether there are 1 + 4 + 7 +4 = 16 possible ordered pairs.



24. D Construct chord BC. Since minor arc AB measures 90°, ∠ACB is inscribed in an arc of 270°, so that m∠ACB = 135°. Therefore, m∠DCB = 45°, making ΔBCD an isosceles right triangle. Letting CD = x, CB = $x\sqrt{2}$. Now construct radius PC. Since ΔAPC is equilateral, m∠CPB = 90 – 60 = 30°. Using the Law of Cosines on ΔPCB



$$2x^2 = 4^2 + 4^2 - 2(4)(4)\cos 30$$

$$2x^2 = 32 - 32\frac{\sqrt{3}}{2} = 32 - 16\sqrt{3}$$

$x^2 = 16 - 8\sqrt{3}$. Since $\left(2\sqrt{3} - 2\right)^2 = 16 - 8\sqrt{3}$, the correct choice is D.

25. B Consider the seven-digit number $axbyczd$, where a, b, c, d, x, y, z are the integers 1, 2, …,7 in some order, each appearing exactly once. Let $P = a + b + c + d$ and $Q = x + y + z$. Then by a well-known divisibility rule for 11, $P - Q$ is divisible by 11. Because $P + Q = 28$ and $P - Q = (P + Q) - 2Q = 28 - 2Q$, we know that $P - Q$ is even and so must be divisible by 22. Since x, y, and z are all distinct,

$$6 = 1 + 2 + 3 \le Q \le 7 + 6 + 5 = 18 \text{ and}$$
$$10 = 1 + 2 + 3 + 4 \le P \le 7 + 6 + 5 + 4 = 22$$

Therefore, $-8 \le P - Q \le 16$. But the only multiple of 22 between -8 and 16 is 0. Thus, $P - Q = 0$ and since we have already shown $P - Q = 28 - 2Q$, we have $P = Q = 14$. (Notice this is true of the example given with the problem 1,235,476 which is divisible by 11.)

There are four ways of expressing 14 as the sum of three distinct integers from the set {1, 2, 3, 4, 5, 6, 7}, namely

$$14 = 7 + 6 + 1 = 7 + 5 + 2 = 7 + 4 + 3 = 6 + 5 + 3.$$

Corresponding to each of these, there are 3! ways of choosing x, y, z and 4! ways of choosing a, b, c, d. Therefore, there are (4)(3!)(4!) = 576 numbers of the required form.