THE 2015–2016 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. Between 1934 and 2015 there were 13 different presidents of the United States and 16 different vice presidents. If 11 of the vice presidents were never president, how many of the presidents were never vice president?

   (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

2. In the addition "cryptarithm" at the right, each letter represents one of the digits from 0 to 9 (different letters represent different digits). What is the smallest possible value for the four digit number FIVE.

$$\begin{array}{r} \mathrm{F\ O\ U\ R} \\ +\ \ \mathrm{O\ N\ E} \\ \hline \mathrm{F\ I\ V\ E} \end{array}$$

   (A) 1345 (B) 1475 (C) 1486 (D) 1627 (E) 1648

3. A group of boys formed a secret club. For dues, each boy paid as many dollars as there were boys in the club. When four more boys joined the club, these four boys each paid as many dollars as there were boys now in the club. After this, no more boys joined, and the club reported $301 in its treasury. How many boys are now in the club?

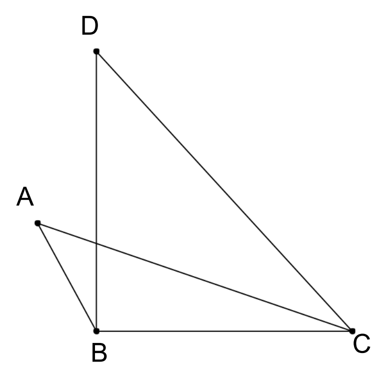
   (A) 15 (B) 16 (C) 17 (D) 18 (E) 19

4. Suppose a six-sided die has sides numbered one through six. If a person throws the die two times, what is the probability that the second number will be larger than the first?

   (A) $\frac{1}{6}$ (B) $\frac{1}{2}$ (C) $\frac{5}{12}$ (D) $\frac{7}{18}$ (E) None of these

5. In the diagram, $\angle ABD \cong \angle DCA$ and $\overline{BD}$ is perpendicular to $\overline{BC}$. If the measure of $\angle DCB$ is 50°, what is the measure of $\angle A$?

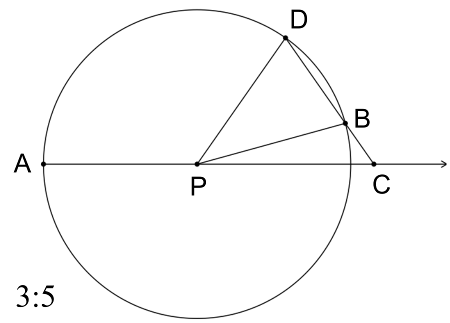
   (A) 25° (B) 30° (C) 40° (D) 45° (E) 50°



6. There are two four-digit numbers, each of the form $\underline{A}\,\underline{B}\,\underline{C}\,\underline{A}$, with the property that the two-digit number $\underline{A}\,\underline{B}$ is a prime, the two-digit number $\underline{B}\,\underline{C}$ is a square, and the two-digit number $\underline{C}\,\underline{A}$ is the product of a prime and a square greater than 1. Compute the sum of these two four-digit numbers.

(A) 10,657 (B) 11,531 (C) 12,185 (D) 13,729 (E) 14,363

7. In the circle shown with center P, radius $\overline{\text{AP}}$ is extended to point C outside the circle and point D is chosen on the circle in such a way that $\overline{\text{DC}}$ intersects the circle at point B and $\overline{\text{DC}}$ is equal in length to the radius of the circle. Compute the ratio of the measure of ∠CPD to the measure of ∠APB.



(A) 1:2 (B) 1:3 (C) 1:4 (D) 2:5 (E) 3:5

8. The *midrange* of a set of numbers is the average of the greatest value and least value in the set. For a set of six increasing, nonnegative integers, the mean, the median, and the midrange are all 5. How many such sets are there?

(A) 8 (B) 9 (C) 10 (D) 12 (E) None of these

9. On Dr. Garner’s last math test, the mean score achieved by 75% of a class was 5 points lower than the mean of the whole class. The mean of the remaining students was how many points above the class mean?

(A) 15 (B) 12 (C) 9 (D) 8 (E) 6

10. The sum of 28 consecutive positive odd integers is a perfect cube. What is the smallest possible first number in such a set of 28 numbers?

(A) 59 (B) 61 (C) 67 (D) 69 (E) 71

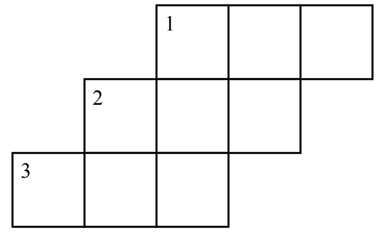
11. Don and Debbie have a total of \$6.07 consisting of pennies, nickels, and quarters. Don has only quarters and nickels, and Debbie has only quarters and pennies. Don has seven times as many nickels as Debbie has pennies. What is the total number of coins that Don and Debbie have?

(A) 93 (B) 97 (C) 103 (D) 105 (E) 109

12. *ABCD* is a square of side length 1. *EFGH* is a square that has one vertex on each side of *ABCD*. If the sides of *EFGH* make an angle θ with the sides of *ABCD*, then the area of *EFGH* is

(A) $\dfrac{1}{4\sin\theta\cos\theta}$ (B) $\dfrac{1}{1+\sin 2\theta}$ (C) $\dfrac{1}{\tan\theta+\cot\theta}$ (D) $\dfrac{1}{2}$ (E) $\dfrac{1}{4\cos^2\theta}$

13. Each integer from 1 to 9 is entered exactly once in the “cross-number” puzzle shown in such a way that the three-digit numbers appearing in 1-across, 2-across, 3-across, and 1-down are perfect squares. Compute the two-digit number appearing in 2-down.



(A) 76 (B) 58 (C) 52 (D) 36 (E) 12

14. The solutions to the equation $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers, compute the value of $\frac{a^2}{b+1}$.

(A) 2 (B) 3 (C) $\frac{16}{5}$ (D) $\frac{5}{2}$ (E) None of these

15. Two sides of a triangle have lengths of 3 inches and 5 inches, and the area of the triangle is 6 square inches. The triangle, however, is not a right triangle. If the number of inches in the length of the third side of this triangle is $\sqrt{k}$, compute k.

(A) 40 (B) 48 (C) 52 (D) 54 (E) 60

16. The first two terms of a geometric sequence are i and $i + 1$ (where $i = \sqrt{-1}$), in that order. Compute the value of the fifteenth term.

(A) -64 (B) $64i + 64$ (C) $-64i - 64$ (D) -128 (E) $128i - 128$

17. If a, b, c are distinct positive integers and $a + b + c = 2015$ and $ab - c = 2015$, compute the value of c.

(A) 529 (B) 729 (C) 1,089 (D) 1,369 (E) 1,849

18. Suppose $a = 2^{(\sqrt{n})!}$, $b = 2^{2^n}$, and $c = (2^n)!$ where n is a perfect square greater than 1,000,000. From largest to smallest, which of the following is the correct order for a, b, and c?

(A) a, b, c (B) b, c, a (C) c, b, a (D) c, a, b (E) b, a, c

19. A parallelogram has sides of length 4 and 6. The length of one of its diagonals is 8. If the length of the other diagonal is $\sqrt{k}$, what is the value of k?

(A) 20 (B) 24 (C) 28 (D) 32 (E) 40

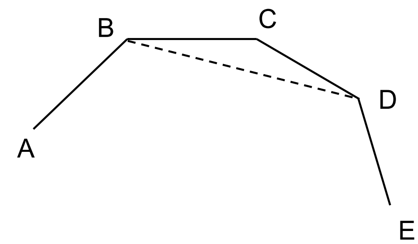
20. If $f(11) = 11$, and for all x, $f(x+3) = \dfrac{f(x)-1}{f(x)+1}$, compute $f(2015)$.

(A) 11 (B) $-\dfrac{1}{11}$ (C) $\dfrac{5}{6}$ (D) $-\dfrac{6}{5}$ (E) -11

21. Let A be a two-digit integer and let B be the integer obtained by reversing the digits of A. If $A^2 - B^2$ is the square of an integer, compute $A^2 + B^2$.

(A) 4,941 (B) 5,265 (C) 5,913 (D) 6,885 (E) 7,361

22. Shown in the accompanying diagram is part of a regular polygon (ABCDE…) of unspecified number of sides $n > 4$. Another regular polygon having $\overline{BD}$ as one side and angles ABD and EDB as consecutive angles is drawn. Which of the following is a possible value of n?



(A) 55 (B) 56 (C) 57 (D) 58 (E) 59

23. What is the ninth digit from the right in the value of 101^{20}?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

24. If $\log_{\sin x}(\tan x) = \log_{\tan x}(\sin x)$, compute the value of $\cos x$.

(A) $\dfrac{-1+\sqrt{2}}{2}$ (B) $\dfrac{1-\sqrt{3}}{2}$ (C) $\dfrac{-1+\sqrt{3}}{2}$ (D) $\dfrac{-1+\sqrt{5}}{2}$ (E) $\dfrac{1-\sqrt{5}}{2}$

25. A triangle has vertices A(0,0), B(3,0), and C(3,4). If ΔABC is rotated counterclockwise around the origin until point C lies on the positive y-axis, compute the area of the region common to the original triangle and the rotated triangle.

(A) $\dfrac{21}{16}$ (B) $\dfrac{25}{16}$ (C) $\dfrac{29}{16}$ (D) $\dfrac{35}{16}$ (E) None of these

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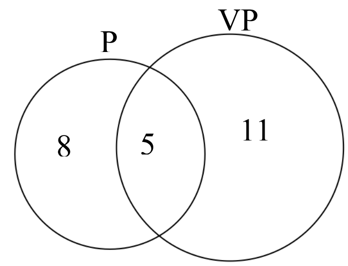
Solutions



1. D Make a Venn diagram. Answer is 8.

The eight were: Franklin Roosevelt, Dwight Eisenhower, John Kennedy, Jimmy Carter, Ronald Reagan, Bill Clinton, George W. Bush, Barack Obama.

2. C Clearly, R = 0. The number F I V E will be smallest if F = 1. This means O must be 2, 3, or 4, and 2 makes the I smallest. Thus, I = 4. This leaves 3 and 5 for U and N (in either order), making V = 8. This leaves only 6, 7, or 9 for E. Therefore, the smallest value for the number F I V E is 1486.

$$\begin{array}{rcccc} & F & O & U & R \\ + & & O & N & E \\ \hline & F & I & V & E \end{array} \qquad \begin{array}{rcccc} & 1 & 2 & 3 & 0 \\ + & & 2 & 5 & 6 \\ \hline & 1 & 4 & 8 & 6 \end{array}$$

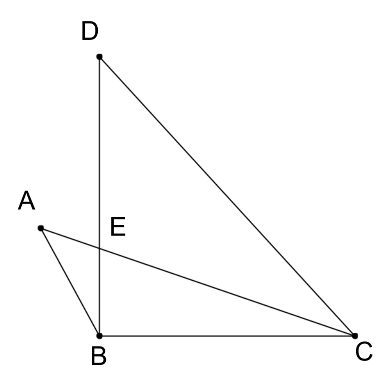
3. E Let n = number of boys in the club at the start. Then $(n)(n) + 4(n + 4) = 301$
From which $n^2 + 4n - 285 = 0$. Factoring, $(n - 15)(n + 19) = 0$, and $n = 15$.
Thus, there are 15 + 4 = 19 boys now in the club.

4. C Method 1: The probability that the second number is the same as the first is 1/6. Therefore, 5/6 of the time, one die has a higher number than the other. By symmetry, the probability that the second die has the higher number is (1/2)(5/6) = 5/12.

Method 2: There is a 1/6 probability that the first die is a one, and in that case, a 5/6 probability that the second die is larger. Therefore, the probability that both occur is (1/6)(5/6). Similarly, if the first die is a 2, 3, 4, or 5, the required probabilities are, respectively, (1/6)(4/6), (1/6)(3/6), (1/6)(2/6), (1/6)(1/6). Therefore, the probability that the second die is larger than the first is

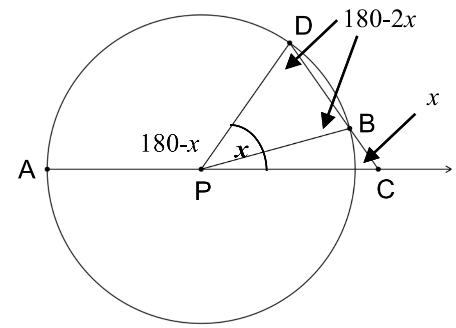
$$(1/6)(5/6) + (1/6)(4/6) + (1/6)(3/6) + (1/6)(2/6) + (1/6)(1/6) = 15/36 = 5/12.$$

5. C In ΔDBC, $m\angle D = 40$. Since $\angle ABD \cong \angle DCA$ and vertical angles DEC and AEB are congruent, the third angles of triangles DEC and AEB must also be congruent. Therefore, $\angle A \cong \angle D$, making $m\angle A = 40$.



6. B Since $\underline{A}\ \underline{B}$ is prime, B must be 1, 3, 7, or 9. Since $\underline{B}\ \underline{C}$ is a square and there are no squares in the 70's or 90's, $B = 1$ or 3, and this means $C = 6$. A quick check of the integers from 61 to 69 shows that 63 = (9)(7) and 68 = (4)(17) satisfy the third condition of the problem. Thus A = 3 or 8. Remembering the first condition, the only 4-digit numbers that work are 3163 and 8368. The required sum is 11,531

7. B Since PD = DC, ΔPDC is isosceles and ∠CPD ≅ ∠PCD. Since ΔDPB is also isosceles, ∠PDB ≅ ∠PBD. Let m∠PCD = x. Representing angle measures as shown in the diagram, m∠BPD = 180 – 2(180 – 2x) = 4x – 180. Therefore, m∠APB = (180 – x) + (4x – 180) = 3x. Hence, the required ratio is 1:3.



8. C It is easy enough to list all 10 possibilities: {0,1,2,8,9,10}, {0,1,3,7,9,10}, {0,1,4,6,9,10} {0,2,3,7,8,10}, {0, 2,4,6,8,10}, {0,3,4,6,7,10}, {1,2,3,7,8,9}, {1,2,4,6,8,9}, {1,3,4,6,7,9}, and {2,3,4,6,7,8}.

9. A Let 3k and k denote the number of students in each subgroup and let M denote the class mean. Then $4kM - 3k(M - 5) = kM + 15k$ represents the total points scored by the smaller subgroup of k students. Therefore, the mean of this group is

$$\frac{kM + 15k}{k} = M + 15.$$

10. E Let S = the sum of the 28 consecutive odd integers. Then

$$S = a + (a + 2) + (a + 4) + \ldots + (a + 54) = \frac{28}{2}(2a + 54) = 28(a + 27).$$

Since $28 = 7(2^2)$, this last expression will be a perfect cube if $(a + 27) = 2(7^2) = 98$. Therefore, $a = 71$.

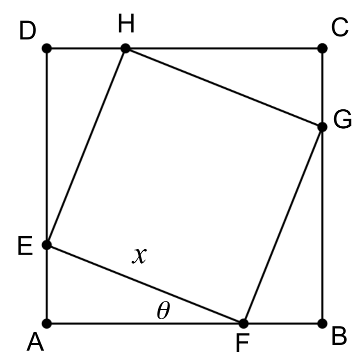
11. C Let P, N, and Q represent the total number of pennies, nickels, and quarters, respectively. Then $P + 5N + 25Q = P + 35P + 25Q = 607$. Therefore, $Q = \frac{607 - 36P}{25}$. Since Q is a positive integer, the numerator of this fraction must be divisible by 25 and so must end in a 5 (it cannot end in 0 since 607 is odd and 36P is even). This can only happen if P ends in 2 or 7. Trying P = 2, 7, and 12, we find only 12 works. If $P \geq 17$, the numerator is negative. Therefore, $P = 12$, $N = 84$, and $Q = 7$ and the total number of coins is 103.

12. B Let EF = x. Then AE = $x \sin\theta$, and DE = AF = $x \cos\theta$. Therefore,

$$x\sin\theta + x\cos\theta = x(\sin\theta + \cos\theta) = 1.$$

Squaring both sides and solving for x^2, we obtain

$$x^2 = \frac{1}{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta} = \frac{1}{1 + 2\sin\theta\cos\theta} = \frac{1}{1 + \sin 2\theta}$$

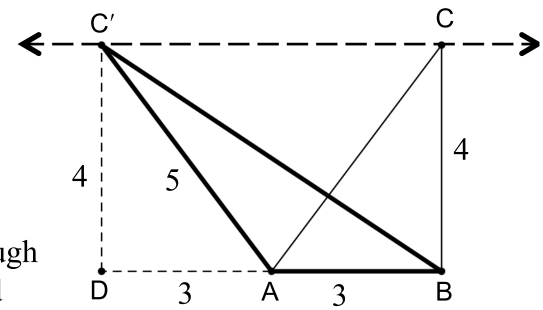


		1 3	6	1
	2 5	2	9	
3 7	8	4		

13. B List all the three digit perfect squares: 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961. Since each digit from 1 to 9 is to be represented exactly once, eliminate all those with duplicate digits or a 0. We are left with
169, 196, 256, 289, 324, 361, 529, 576, 625, 729, 784, 841, 961.

Only two of the numbers contain the digit 3, namely 324 and 361. If we choose 324 as one of the "across" entries, then the other two cannot contain the digits 3, 2, and 4. This leaves only 169, 196, 576, and 961. But then it is impossible to have the digit 8. Therefore we must choose 361 instead. Eliminating the other numbers which contain a 3, 6, or 1, we are left with 289, 529, 729, and 784. From these we must choose 529 and 784 because they are the only occurrences of 5 and 4, respectively. Only the arrangement shown gives us a perfect square 324 for 1-down. Therefore, 2-down is 58.

14. B Represent the solutions with n, $n+1$, and $n+2$. Then $a = -[n + (n+1) + (n+2)]$ and $a^2 = (3n+3)^2 = 9(n+1)^2$. Also, $b = n(n+1) + n(n+2) + (n+1)(n+2) = 3n^2 + 6n + 2$, so that $b + 1 = 3n^2 + 6n + 3 = 3(n+1)^2$. Therefore, $\dfrac{a^2}{b+1} = \dfrac{9(n+1)^2}{3(n+1)^2} = 3$.



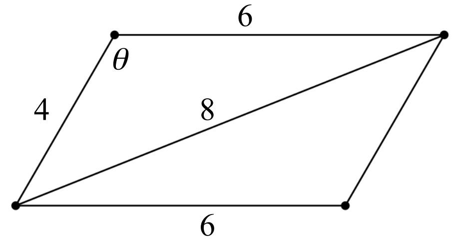
15. C Since the area of the triangle is 6, the altitude to the 3″ side must have length 4″. In the diagram, ΔABC illustrates the 3-4-5 right triangle that fulfills the requirement. To find the other triangle, extend $\overline{BA}$ through point A to a point D that is 3″ from A. Let C′ be the point on the parallel to $\overline{BA}$ through point C that is 4″ from D, as shown. ΔAC′B has the required properties. Using the Pythagorean Theorem on ΔBC′D, C′B = $\sqrt{52}$.

16. D The ratio of the geometric progression is $\dfrac{i+1}{i} = 1 - i$. Therefore, the third term is $(i+1)(1-i) = 2$. The fourth term is $2(1-i) = 2 - 2i$, the fifth term is $(1-i)(2-2i) = -4i$, and the sixth term is $(1-i)(-4i) = -4i - 4 = -4(1+i)$. Therefore, the sixth term is -4 times the second term. Thus the tenth term is $16(1+i)$ and the fourteenth term is $-64(1+i)$. Multiplying this by $(1-i)$ gives -128 as the fifteenth term.

17. E Adding the given equations gives $a + b + ab = 4030$. Adding one to both sides, we get $a + b + ab + 1 = (a+1)(b+1) = 4031$. Since $4031 = (29)(139)$, and both 29 and 139 are prime, $a = 28$ and $b = 138$ (or vice versa, since a and b are interchangeable in this situation). Therefore $a + b + c = 28 + 138 + c = 2015$, and $c = 1849$.

18. C The answer is c, b, a. Clearly, $k! > 2^k$ for all integers $k > 3$, which implies that $c > b$. To see that $b > a$, we need to show that $2^n > \left(\sqrt{n}\right)!$. Let $n = m^2$. Then we wish to show that $2^{m^2} > m!$. This is certainly true because $2^{m^2} = \left(2^m\right)^m$ and $2^m > m$ for $m > 2$.

19. E Using the Law of Cosines on the triangle with sides 4, 6, 8,



$$8^2 = 4^2 + 6^2 - 2(4)(6)\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{4}$$

Since the consecutive angles of a parallelogram are supplementary, the other angle of the parallelogram is $(180 - \theta)°$. Let d represent the length of the other diagonal Now using the Law of Cosines on the triangle with sides 4, 6, d,

$$d^2 = 4^2 + 6^2 - 2(4)(6)\left[\cos(180 - \theta)\right] = 52 - 48(-\cos\theta) = 52 - 48\left(\frac{1}{4}\right) = 40 \text{ and } d = \sqrt{40}.$$

20. A $f(11) = 11$, $f(14) = \dfrac{f(11)-1}{f(11)+1} = \dfrac{5}{6}$, $f(17) = \dfrac{f(14)-1}{f(14)+1} = -\dfrac{1}{11}$,

$f(20) = \dfrac{f(17)-1}{f(17)+1} = -\dfrac{6}{5}$ and $f(23) = \dfrac{f(20)-1}{f(20)+1} = 11$.

Therefore, for all positive integers n, $f(11) = f(23) = f(35) = \ldots = f(11+12n)$.
Since $f(2015) = f(11 + 12\cdot 167)$, then $f(2015) = 11$.

21. E Let $A = 10x + y$ and $B = 10y + x$, where $x = 1, 2, 3 \ldots, 9$, and $y = 0, 1, 2, 3 \ldots, 9$. Then

$$A^2 - B^2 = (10x + y)^2 - (10y + x)^2 = 99x^2 - 99y^2 = (9)(11)(x + y)(x - y)$$

Since this must be a perfect square and $x - y < 10$, 11 must be a factor of $x + y$. But $x + y \le 17$, so $x + y = 11$. Therefore, $x - y$ is a perfect square. Hence, $x - y = 1, 4$, or 9. Since x +y and x – y are either both even or both odd, the possible combinations are:

(i) $x + y = 11$ and $x - y = 1$ and (ii) $x + y = 11$ and $x - y = 9$.

(i) yields $x = 6$, $y = 5$ from which $A = 65$. (ii) yields $x = 10$, $y = 1$ which is not acceptable. Therefore, $A = 65$ and $A^2 + B^2 = 65^2 + 56^2 = 7{,}361$.

22. C The measure of $\angle C$ is $\dfrac{180(n-2)}{n}$. The measure of $\angle CBD$ is $\dfrac{180 - \dfrac{180(n-2)}{n}}{2} = \dfrac{180}{n}$.

The measure of $\angle DBA$ is $\dfrac{180(n-2)}{n} - \dfrac{180}{n} = \dfrac{180n - 540}{n}$

For the second polygon to be a regular polygon with m sides, $\dfrac{180(m-2)}{m} = \dfrac{180n - 540}{n}$.

Simplifying this last equation gives $m = \dfrac{2}{3}n$. Since m is an integer, any value of $n > 4$ which is a multiple of 3 will suffice. The only choice that is a multiple of 3 is 57.

23. E Expand 101^{20} using the binomial theorem:

$$(100+1)^{20} = 1 + (20)(100) + \binom{20}{2}(100^2) + \binom{20}{3}(100^3) + \binom{20}{4}(100^4) + \ldots$$

The first 3 terms will not affect the ninth digit from the right since each has fewer than 8 digits. None of the omitted terms at the end will affect the ninth digit from right since each has more than 9 terminal zeros. The fourth and fifth terms are 1,140,000,000 and 484,500,000,000, so that the ninth digit from the right in their sum is 5 + 1 = 6.

24. D $\log_{\sin x}(\tan x) = \log_{\tan x}(\sin x) = \dfrac{1}{\log_{\sin x}(\tan x)}$ using the change of base formula,

where $\sin x > 0$ and $\tan x > 0$.

Therefore, $\left[\log_{\sin x}(\tan x)\right]^2 = 1$, which implies $\log_{\sin x}(\tan x) = \pm 1$.

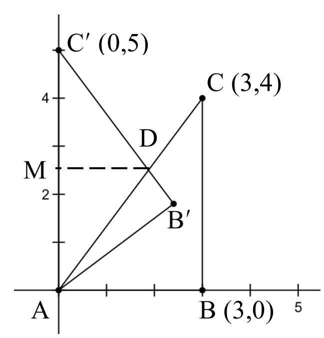
$\log_{\sin x}(\tan x) = 1 \quad \Rightarrow \quad \tan x = \sin x \quad \Rightarrow \quad \cos x = 1$. However, this would make $\sin x = 0$.

$\log_{\sin x}(\tan x) = -1 \quad \Rightarrow \quad \cos x = \sin^2 x = 1 - \cos^2 x$

Therefore, $\cos^2 x + \cos x - 1 = 0 \quad \Rightarrow \quad \cos x = \dfrac{-1 \pm \sqrt{5}}{2}$.

Since $\dfrac{-1-\sqrt{5}}{2} < -1$, the only possible value of $\cos x$ is $\dfrac{-1+\sqrt{5}}{2}$.

25. A Since ABC is a right triangle, AC = AC′ = 5. Let D be the point of intersection of $\overline{AC}$ and $\overline{B'C'}$, as shown in the diagram. Since BC is parallel to the y-axis, $\angle C \cong \angle CAC'$. Therefore, $\angle B'C'A \cong \angle CAC'$, so that $\Delta ADC'$ is isosceles. Thus, the altitude from D meets $\overline{AC'}$ at its midpoint, M, and C′M = 2.5.



Since $\Delta MDC'$ and ΔBAC are both right triangles with $\angle C \cong \angle C'$, the triangles are similar. Thus, $\dfrac{MC'}{BC} = \dfrac{C'D}{AC}$. Substituting, $\dfrac{2.5}{4} = \dfrac{C'D}{5}$, from which $C'D = \dfrac{25}{8}$.

Hence, $DB' = 4 - \dfrac{25}{8} = \dfrac{7}{8}$, and the area of right triangle DAB′ is

$$\frac{1}{2}(AB')(B'D) = \frac{1}{2}(3)\left(\frac{7}{8}\right) = \frac{21}{16}$$