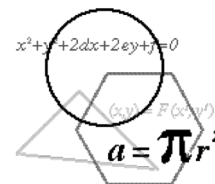**THE 2016–2017 KENNESAW STATE UNIVERSITY**
HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. The average of ten numbers is 190. If one of the numbers is doubled, the new average is 500. What is the value of the number that was doubled?

    (A) 310   (B) 380   (C) 690   (D) 3100   (E) 3800

2. If the circumference of a circle is decreased by 20%, by what percent is the area of the circle decreased?

    (A) 20   (B) 32   (C) 36   (D) 40   (E) 64

3. Three students took a six-question true-false exam. Debbie answered #1 and #2 true and the rest false, Don answered #2 and #3 true and the rest false, and Chuck answered #3 and #4 true and the rest false. If Debbie and Don each got five of the questions correct, what is the largest possible number of correct answers that Chuck could have?

    (A) 1   (B) 2   (C) 3   (D) 4   (E) 5

4. Mary wanted to complete the 4x4 grid shown by filling in the individual squares so that each row and each column contained each of the numbers 1, 2, 3, 4 exactly once. What is the value of x?

    (A) 1   (B) 2   (C) 3   (D) 4   (E) It cannot be done

5. The nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9 are written in a straight line in a certain order so that:
    (i) if any two digits add up to 10 they are written adjacent to each other, and
    (ii) if any two digits add up to 9 they are written adjacent to each other.

    The second digit from the left is odd. What is the seventh digit from the left?

    (A) 2   (B) 3   (C) 4   (D) 6   (E) 8

6. A square is plotted on a coordinate axis system. When the four x-coordinates and the four y-coordinates of the square's vertices are added, the sum is 348. If the coordinates of the center of the square are $(x, 37)$, compute the value of x.

(A) 30   (B) 35   (C) 40   (D) 45   (E) 50

7. Let $f(x) = ax^7 + bx^3 + cx - 5$. If $f(-7) = 7$, what is the value of $f(7)$?

(A) -17   (B) -12   (C) -7   (D) 12   (E) 17

8. A palindrome is a number which reads the same forwards and backwards (for example 383 and 15051 are palindromes). Compute the <u>sum</u> of all possible two digit numbers a such that $a + \dfrac{1}{a}$, when expressed as an improper fraction in lowest terms, has a numerator which is a three digit palindrome.

(A) 30   (B) 35   (C) 40   (D) 45   (E) 50

9. A semicircle of diameter two inches is cut from a piece of paper. The semicircle is folded so that the midpoint of minor arc AB is tangent to the semicircle's diameter at its center, as shown. Which of the following is closest to the number of inches in the length of the fold (segment $\overline{AB}$ in the diagram)?
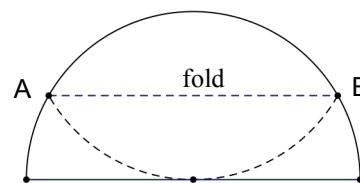
(A) 1.4   (B) 1.5   (C) 1.6   (D) 1.7   (E) 1.8

10. Three distinct digits are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and then the three digits are multiplied together to form a number. What is the probability that this number is a multiple of 10?

(A) $\dfrac{1}{12}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{4}{21}$ (D) $\dfrac{9}{32}$ (E) $\dfrac{11}{42}$

11. Find the sum of all positive numbers x such that $x^{x\sqrt{x}} = (x\sqrt{x})^x$.

(A) 1   (B) 1.5   (C) 2.25   (D) 2.5   (E) 3.25

12. Find the smallest positive integer N which can be expressed as the sum of 11 consecutive positive integers and as the sum of 7 consecutive positive even integers and as the sum of 5 consecutive positive multiples of 3.

(A) 1122   (B) 1956   (C) 1920   (D) 2310   (E) None of these

13. How many values of x, $0° \le x < 360°$, satisfy the equation $\sec x + \tan x + \cos x = 0$?

    (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

14. Dr. Garner asked his math students to find the sum of all three terms of a geometric sequence in which the middle number was missing. Unfortunately, one student confused geometric sequences with arithmetic sequences, and then completed the problem with no other mistakes, obtaining an answer of 351. If all the numbers used by Dr. Garner and the student were distinct positive integers, and if the ratio of the geometric sequence was an integer, what was the correct answer to Dr. Garner's problem?

    (A) 248 (B) 279 (C) 294 (D) 343 (E) 387

15. The sum of the 3-digit numbers $\underline{3}\,\underline{5}\,\underline{x}$ and $\underline{4}\,\underline{y}\,\underline{7}$ is divisible by 36. Compute the smallest possible sum $x + y$.

    (A) 6 (B) 8 (C) 9 (D) 10 (E) 12

16. Let $N = (k + 1) + (k + 2) + \ldots + (k + 31)$, where k is a positive integer. Compute the sum of the two smallest values of k for which N is a perfect square.

    (A) 62 (B) 108 (C) 123 (D) 141 (E) None of these

17. Given two regular polygons with m and n sides, $m > n$, such that the degree measure of each interior angle of each polygon is a multiple of 7. All the vertices of the n-sided polygon are vertices of the m-sided polygon. Compute $m + n$.

    (A) 25 (B) 39 (C) 53 (D) 60 (E) 81

18. The three-digit base b number $\underline{A}\,\underline{B}\,\underline{C}_b$ equals 169_{10}. If A, B, and C are distinct, and $A \ne 0$, determine the number of possible positive integral bases b.

    (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

19. Given the polynomial $f(x) = ax^3 + 2016x^2 + c$, with a and c nonzero. If c is a root of $f(x) = 0$, which of the following must represent the sum of the remaining two roots?

    (A) $\dfrac{a}{c}$ (B) $\dfrac{-2016}{c}$ (C) $\dfrac{1}{ac}$ (D) $\dfrac{2016}{ac}$ (E) $\dfrac{ac}{2016}$

20. For all possible values of a fixed constant a, the system $x + ay = 2016$ and $ax + y = 2016a + 3$ has exactly three integral pairs of solutions, (x_1, y_1), (x_2, y_2), and (x_3, y_3). Compute the value of $x_1 y_1 + x_2 y_2 + x_3 y_3$.

(A) 1008   (B) 2016   (C) 3024   (D) 4032   (E) None of these

21 In triangle ABC, AB = 7, BC = 19, and AC = 24. A circle of radius 5 is constructed with center B, intersecting $\overline{AC}$ at points D and D'. Compute the measure of $\angle DBD'$.

(A) 30°   (B) 45°   (C) 60°   (D) 75°   (E) 90°

22. The expansions of $(1 + x)^n$ and $(1 + x)^{n+1}$ are each written in <u>ascending</u> powers of x. The ratio of the fourth term of the expansion of $(1 + x)^n$ to the fifth term of the expansion of $(1 + x)^{n+1}$ is $\dfrac{2}{3x}$. If n ≥ 3, what is the value of n?

(A) 5   (B) 6   (C) 7   (D) 8   (E) 9

23. Let S be the sum of the base 10 logarithms of all of the proper divisors of 1,000,000. (By proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself.) What is the integer nearest to S?

(A) 100   (B) 123   (C) 141   (D) 147   (E) 150

24. A square is constructed so that its sides are parallel to the coordinate axes. When lines with slopes $\dfrac{1}{3}$ and $\dfrac{2}{3}$ are drawn through the center of the square, it is divided into two congruent pentagons and two congruent triangles. Compute the ratio of the area of one of the pentagons to the area of one of the triangles.

(A) $\dfrac{8}{1}$ (B) $\dfrac{9}{1}$ (C) $\dfrac{10}{1}$ (D) $\dfrac{11}{1}$ (E) $\dfrac{12}{1}$

25. The standard deviation, σ, of a set of N numbers is a measure of "spread" from the mean, μ. It is given by the formula $\sigma = \sqrt{\dfrac{\sum (x_i - \mu)^2}{N}}$, where x_i represent the individual numbers in the set ($i = 1, 2, 3, \ldots, N$). A statistics student taking a test on standard deviation made an unfortunate error. Copying a set of 20 numbers, he accidentally left off one of the numbers, a 28. Thus he was only aware of 19 numbers. As a result, his value of μ was 8 instead of the correct mean. He made no other errors and, amazingly, he obtained the standard deviation for the original set! If this standard deviation is $\sqrt{k}$, compute k.

(A) 380   (B) 416   (C) 468   (D) 520   (E) None of these

Solutions:

1. D Let the ten original numbers be represented as $a_1, a_2, a_3, \ldots, a_{10}$. Then

$$\frac{a_1 + a_2 + a_3 + \ldots + a_{10}}{10} = 190 \quad \text{and} \quad a_1 + a_2 + a_3 + \ldots + a_{10} = 10(190) = 1900.$$

Similarly, $2a_1 + a_2 + a_3 + \ldots + a_{10} = 10(500) = 5000$. Subtracting the last two equations, we obtain $a_1 = 3100$.

2. C Since the circumference of a circle is directly proportional to the radius, the radius is also decreased by 20%. Therefore, the new area is A = $\pi\left(\frac{4}{5}r\right)^2 = \frac{64}{100}\pi r^2$. Thus the area is decreased by 36%.

3. C Since Debbie and Don each got five of the questions correct, there are two possibilities.
    (i)   Questions 1, 2, and 3 are true, and questions 4, 5, 6 are false.
    (ii)   Question 2 is true and questions 1, 3, 4, 5, 6 are false.
In (i) Chuck has three correct answers (questions 3, 5, and 6). In (ii) Chuck also has three correct answers, (questions 1, 5, 6). Either way, the most possible correct for Chuck is 3.

4. D First fill in the bottom row, then the second column, then the top row, then the third column. Thus $x = 4$.

2	4	3	1
	2	1	
	1	x	
1	3	2	4

5. D From (a) the following pairs of digits must be adjacent: 1 and 9; 2 and 8; 3 and 7; 4 and 6.
From (b) the following pairs of digits must be adjacent: 1 and 8; 2 and 7; 3 and 6; 4 and 5.
Each digit except 5 and 9 has two required adjacent digits and, therefore, cannot be at either end.
So, the end digits must be 5 and 9. There are then only two possible arrangements:
          546372819 and 918273645.
Since the second digit from the left must be odd, the only possibility is the second number above. Therefore, the seventh digit from the left is 6.

6. E The coordinates of the center of the square are the average of the coordinates of either diagonal's endpoints. Therefore, the sum of the four y-coordinates of the vertices is (4)(37) = 148. Then the sum of the remaining x-coordinates is 348 - 148 = 200. The x-coordinate of the center of the square is 200÷4 = **50**.

7. A Since $f(x) = ax^7 + bx^3 + cx - 5$, then $f(-x) = -ax^7 - bx^3 - cx - 5$ Therefore, $f(x) + f(-x) = -10$. Since $f(-7) = 7$, $f(7) = -17$.

8. **B** $a + \dfrac{1}{a} = \dfrac{a^2+1}{a}$. Since $32^2 = 1024$, which is more than three digits, $10 \leq a \leq 31$. The answers can now be obtained by inspection: $10 + \dfrac{1}{10} = \dfrac{101}{10}$ and $25 + \dfrac{1}{25} = \dfrac{626}{25}$ and no others will work. Therefore, $a = 10$ or 25 and the desired sum is 35.

9. **D** Construct radius $\overline{PQ}$ and segment $\overline{QA}$. Then $QA = PA = PQ = 1$. Therefore, $\triangle APQ$ is equilateral with side length 1. The length of the altitude of the triangle is $\dfrac{\sqrt{3}}{2}$. Since $\overline{AB}$ is twice the length of the altitude, $AB = \sqrt{3} \approx 1.732$.
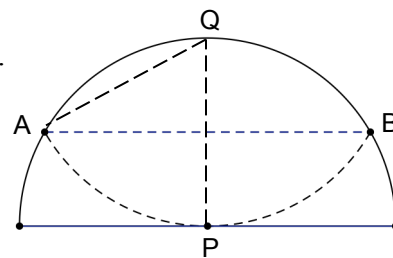
10. **E** One of the three digits must be a 5. At least one of the remaining two digits must be even. If both are even, there are $_4C_2 = 6$ choices. If one is even and the other odd, there are $_4C_1 \cdot _4C_1 = 16$ choices. Therefore, there are 22 numbers divisible by 10. Since there are $_9C_3 = 84$ ways of choosing three digits, the desired probability is $\dfrac{22}{84} = \dfrac{11}{42}$.

11. **E** Clearly, $x = 1$ is a solution. The given equation can be rewritten $x^{x\sqrt{x}} = (x^{\frac{3}{2}})^x$. Thus, if $x \neq 1$, then $x\sqrt{x} = \dfrac{3}{2}x$. Squaring both sides and clearing fractions, $4x^3 - 9x^2 = 0$.

Factoring, we obtain $x^2(4x - 9) = 0 \Rightarrow x = \dfrac{9}{4}$. The desired sum is $1 + \dfrac{9}{4} = \dfrac{13}{4} = 3.25$.

12. **D** Represent the 11 consecutive positive integers as $a-5$, $a-4$, …, $a-1$, a, $a+1$, $a+2$,…, $a+5$. Represent the 7 consecutive even integers as $2b-6$, $2b-4$, $2b-2$, $2b$, $2b+2$, $2b+4$, $2b+6$. Represent the 5 consecutive multiples of 3 as $3c-6$, $3c-3$, $3c$, $3c+3$, $3c+6$. Then $11a = 14b = 15c$. Thus, the required number is the least common multiple of 11, 14, and 15. Since these three numbers are relatively prime, the LCM is $(11)(14)(15) = 2310$.

13. **A** Note that $x \neq 90°, 270°$. Then $\sec x + \tan x + \cos x = \dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x} + \cos x = 0$. Multiplying by $\cos x$, $1 + \sin x + \cos^2 x = 0 \Rightarrow 1 + \sin x + (1 - \sin^2 x) = \sin^2 x - \sin x - 2 = 0$. Factoring, $(\sin x - 2)(\sin x + 1) = 0$. This equation has only one solution, $x = 270°$, which has already been eliminated. Hence the number of solutions is 0.

14. **B** Let the geometric sequence be a, ar, ar^2. Then the student's sequence was a, $\dfrac{1}{2}(a + ar^2)$, ar^2. Therefore, $a + \dfrac{1}{2}(a + ar^2) + ar^2 = 351 \Rightarrow 3a + 3ar^2 = 702$ from which $a = \dfrac{234}{1+r^2}$. This last equation only gives integer values of a if $r = 1$ or 5. Since all the numbers are distinct, $r = 5$ and $a = 9$. Therefore, the three integers from Dr. Garner's sequence are 9, 45, and 225 with a sum of 279.

15. **B** We know that $350 + x + 407 + 10y = 757 + x + 10y = 36k$ for some positive integer k. Since $757 = 36(21) + 1$, then $10y + x = 36m - 1$ for some positive integer m. Since $0 < 10y + x \leq 99$, $10y + x = 35$ or 71. In either case, $x + y = 8$.

16. **C** $(k+1)+(k+2)+\ldots+(k+31)=31k+(1+2+\ldots+31)=31k+\dfrac{(31)(32)}{2}=$

$31k+(16)(31)=31(k+16)$. Since 31 is a prime number, the smallest values of k for which this last expression is a perfect square are values of k such that $k+16=31$ and $k+16=(4)(31)$. From this, $k=15$ and 108, with a sum of 123.

17. **E** Because the formula for the measure of the interior angle of a regular polygon is $\dfrac{(n-2)180}{n}$,

$n-2$ must be a multiple of 7. The first value of n which gives an integer value for the measure of the angle is $n=9$. Trying other values of n (16, 23, 30, 37, 44, 51, 58, 65, 72, …), an integer value results only for $n=30$ and 72. However, because the polygons are regular, all the vertices of the n–sided polygon must be "evenly spaced" at the vertices of the m–sided polygon. Thus, m must be a multiple of n. Only $m=72$ works, so the required ordered pair is $(72, 9)$ and $m+n=81$.

18. **A** $\underline{A}\,\underline{B}\,\underline{C}_b=Ab^2+Bb+C$. If $b=5$, we have $25A+5B+C$, which cannot equal 169 since $A,B,C\le 4$. If $b=6$, $36A+6B+C=169$ for $A=4$, $B=4$, and $C=1$. But this is not an option since A, B, and C must be distinct. If $b=13$, obviously $A=1$ and $B=C=0$, and again this is not an option since A, B, and C are distinct. If $b=14$, then $196A$ is too large. Now try integers values of b from 7 to 12.

    1) For $b=7$, $A=B=3$ and $C=1$ which is not an option since A, B, and C are distinct.
    2) For $b=8$, $A=2$, $B=5$ and $C=1$ works.
    3) For $b=9$, $A=2$, $B=0$ and $C=7$ works.
    4) For $b=10$, $A=1$, $B=6$ and $C=9$ works.
    5) For $b=11$, $A=1$, $B=C=4$, which is not an option since A, B, and C are distinct.
    6) For $b=12$, $A=1$, $B=2$ and $C=1$, which is not an option.

Therefore, the total number of values for b is 3.

19. **C** Let the three roots of $ax^3+2016x^2+c=0$ be c, r_1 and r_2. The sum of all three roots is $\dfrac{-2016}{a}$, so the sum of the two remaining roots is $r_1+r_2=\dfrac{-2016}{a}-c$. However, this is not among the choices, so let's try something else! Since the coefficient of x is 0, we know that (1) $cr_1+cr_2+r_1r_2=0$. We also know that the product of all three roots is $\dfrac{-c}{a}$.

Therefore, $cr_1r_2=\dfrac{-c}{a}$ or (2) $r_1r_2=\dfrac{-1}{a}$. Substituting (2) into (1), $c(r_1+r_2)=\dfrac{1}{a}$, so that the sum of the remaining roots is $\dfrac{1}{ac}$.

20. **B** Multiply the second equation by a and subtract the first equation from the result.

$\begin{aligned}x+ay&=2016\\a^2x+ay&=2016a^2+3a\end{aligned}$ $\longrightarrow$ $(a^2-1)x=2016(a^2-1)+3a$ $\longrightarrow$ $x=2016+\dfrac{3a}{a^2-1}$

Similarly, we can obtain $y=\dfrac{-3}{a^2-1}$. Since x and y must be integers, a^2-1 must evenly divide 3, which only happens if $a=2, 0, -2$. These values of a produce, in order, the ordered pairs $(2018, -1)$, $(2016, 3)$, and $(2014, -1)$. Then $x_1y_1+x_2y+x_3y_3=$
$-2018+6048-2014=2016$.
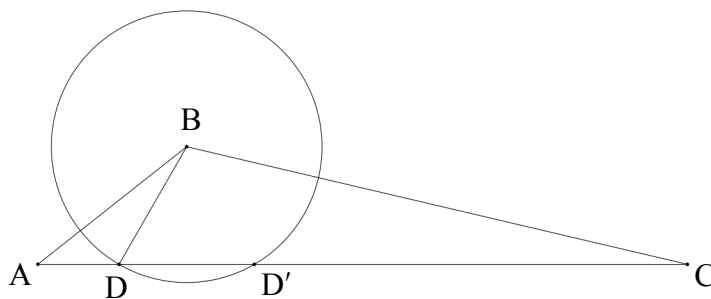
21. **C** Using the Law of Cosines on $\triangle ABC$,

$$19^2 = 7^2 + 24^2 - 2(7)(24)\cos A$$

from which $\cos A = \dfrac{11}{14}$.

Using the Law of Cosines on $\triangle ABD$, and representing AD as x,

$$5^2 = 7^2 + x^2 - 2(7)(x)\left(\frac{11}{14}\right).$$

Simplifying, we obtain $x^2 - 11x + 24 = 0$, from which $x = 3$ and 8. Thus AD $=3$, AD$' = 8$ and the distance between D and D$'$ is 5, making $\triangle$BDD$'$ equilateral. Therefore, the measure of $\angle$DBD$'$ is 60°.

22. **A** The fourth term of the expansion of $(1 + x)^n$ is $_nC_3x^3 = \dfrac{n!}{3!(n-3)!}x^3$. The fifth term of the

expansion of $(1 + x)^{n+1}$ is $_{(n+1)}C_4 x^4 = \dfrac{(n+1)!}{4!(n+1-4)!}x^4 = \dfrac{(n+1)!}{4!(n-3)!}x^4$. The ratio of the two

terms is $\dfrac{\dfrac{n!}{3!(n-3)!}x^3}{\dfrac{(n+1)!}{4!(n-3)!}x^4} = \dfrac{4}{(n+1)x}$. Therefore, $\dfrac{4}{(n+1)x} = \dfrac{2}{3x}$ and $n = 5$.

23. **C** The number $1{,}000{,}000 = (2^6)(5^6)$ has $(6+1)(6+1) = 49$ distinct positive divisors. Apart from 1000, the other 48 divisors form 24 pairs such that the product of each pair is 1,000,000. Since one of these pairs consists of the improper divisors 1 and 1,000,000, it follows that the product of all proper divisors of 1,000,000 is $(1000)(1{,}000{,}000^{23})$, or 10^{141}. Since the sum, S, of the logarithms is equal to the logarithm of the product of these numbers, $S = 141$.

24. **D** Any line passing through the center of a rectangle divides the rectangle into two regions of equal area. Represent the length of the side of the square as $6a$. Then the total area of one of the pentagons and one of the triangles is $\dfrac{1}{2}(6a)(6a){=}18a^2$. The

length of the altitude of each triangle, drawn from the center of the square is 3a. The two given lines intersect a side of the square at a distance of 2a and 1a from a vertex of the square. Therefore, the base of each

triangle has length a. Hence, the area of each triangle is $\dfrac{1}{2}(a)(3a){=}\dfrac{3}{2}a^2$, and the area of

each pentagon is $18a^2 - \dfrac{3}{2}a^2 = \dfrac{33}{2}a^2$. Therefore, the desired ratio is $\dfrac{\dfrac{33}{2}a^2}{\dfrac{3}{2}a^2} = \dfrac{11}{1}$.
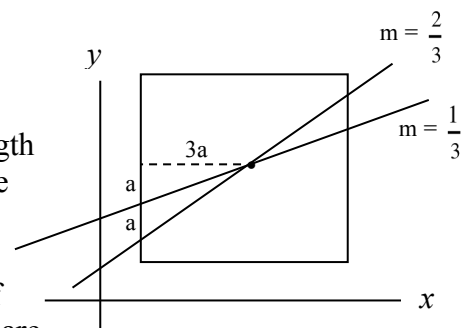
25. **A** Since the sum of the 20 original numbers in $8(19) + 28 = 180$, the mean of all 20 numbers is $180/20 = 9$.

$$\sigma = \sqrt{\frac{\sum(x_i - \mu)^2}{N}} = \sqrt{\frac{\sum(x_i - 9)^2}{20}} = \sqrt{\frac{\sum(x_j - 8)^2}{19}}.$$ Note that $\sum x_i = 9(20) = 180$ and

$\sum x_j = 8(19) = 152$.

$i = 1, 2, 3, \ldots, 20 \qquad j = 1, 2, 3, \ldots, 19$

$$\sqrt{\frac{\sum(x_i - 9)^2}{20}} = \sqrt{\frac{\sum(x_i^2 - 18x_i + 81)}{20}} = \sqrt{\frac{\sum(x_j - 8)^2}{19}} = \sqrt{\frac{\sum(x_j^2 - 16x_j + 64)}{19}}$$

$$\sqrt{\frac{\sum x_i^2 - 18\sum x_i + \sum 81}{20}} \qquad\qquad \sqrt{\frac{\sum x_j^2 - 16\sum x_j + \sum 64}{19}}$$

$$\sqrt{\frac{\sum x_i^2 - 18(180) + 20(81)}{20}} \qquad\qquad \sqrt{\frac{\sum x_j^2 - 16(152) + 19(64)}{19}}$$

Therefore, $$\sqrt{\frac{\sum x_i^2 - 1620}{20}} = \sqrt{\frac{\sum x_j^2 - 1216}{19}}$$

Clearing fractions, $19\sum x_i^2 - 30780 = 20\sum x_j^2 - 24320 \Rightarrow 19\sum x_i^2 = 20\sum x_j^2 + 6460$

But $\sum x_i^2 - \sum x_j^2 = 28^2 = 784$.

$19\sum x_i^2 - 20\sum x_j^2 = 19\left(\sum x_i^2 - \sum x_j^2\right) - \sum x_j^2 = 6460.$

Therefore, $19(784) - \sum x_j^2 = 6460 \Rightarrow \sum x_j^2 = 8436$.

Thus $\sigma = \sqrt{\frac{\sum x_j^2 - 1216}{19}} = \sqrt{\frac{8436 - 1216}{19}} = \sqrt{\frac{7220}{19}} = \sqrt{380}$, so that $k = 380$.