Kennesaw State University
College of Science and Mathematics
Department of Mathematics

THE 2016-2017 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION
PART II

Calculators are NOT permitted

Time allowed: 2 hours

1. Let x, y, and A all be positive integers with $x \neq y$.

   a) Prove that there are infinitely many ordered triples (x, y, A) for which $x^3 + y^3 = A^2$.

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   b) If x, y, and A are all less than 30, find <u>one</u> ordered triple (x, y, A) for which $x^3 - y^3 = A^2$.

2. In Mr. Smith's garden, exactly 1/3 of the flowers are red, 1/3 of the flowers are blue, and 1/3 of the flowers are yellow. Mr. Smith, being a mathematician, discovered that the probability that a randomly picked bunch of three flowers contains exactly 1 red flower is greater than one-half. The same is true for the other two colors as well. Compute, with proof, the largest number of flowers that could be in Mr. Smith's garden.

3. Consider the polynomial $P(x) = px^2 - qx + p$, where p and q are prime numbers. Find, with proof, all possible ordered pairs (p, q) such that the equation $P(x) = 0$ has rational solutions.

4. Let $\langle x \rangle$ denote the fractional part of the real number x so that, for example, $\left\langle \frac{12}{5} \right\rangle = \frac{2}{5}$, $\langle 3 \rangle = 0$, and $\langle \pi \rangle = \pi - 3$. Find, with proof, the smallest positive real number x, larger than 1, such that $\langle x \rangle + \left\langle \frac{1}{x} \right\rangle = 1$.

5. Triangle ABC has side lengths a, b, and c, where a, b, and c are consecutive integers with $a < b < c$. A median drawn to one side of ΔABC divides ΔABC into two triangles, at least one of which is isosceles. Compute, with proof, all possible ordered triples (a, b, c).

Solutions

1. a) Multiplying 9 by any number of the form r^{2k}, where r is a positive integer will produce a perfect square $(3 \cdot r^k)^2$. Let k be a multiple of 3. Then $k = 3m$, where m is a positive integer, and

$$9(r^{2k}) = (1+2^3)(r^{2k}) = r^{2k} + 2^3 r^{2k} = r^{6m} + 2^3 r^{6m} = (r^{2m})^3 + (2r^{2m})^3$$

Therefore, the ordered triple $(x, y, A) = (r^{2m},\ 2r^{2m},\ 3 \cdot r^{3m})$ satisfies the given equation for all positive integers m. More specifically, if $m = 1$, the ordered triple $(r^2,\ 2r^2,\ 3 \cdot r^3)$ satisfies the given equation. Thus there are infinitely many such ordered triples.

b) Factoring, we obtain $x^3 - y^3 = (x-y)(x^2 + xy + y^2) = A^2$. Let $x - y = 1$. Then $(x^2 + xy + y^2) = A^2$. Substituting $y = x - 1$ and simplifying, we obtain $3x^2 - 3x + 1 = A^2 \;\Rightarrow\; 3x(x-1) + 1 = A^2$. Thus we are looking for two consecutive positive integers (if any exist) whose product, when multiplied by 3, is one less than a perfect square. Since x, y, and A are all less than 30, a little trial and error shows that when $x = 8$, $A^2 = 3(8)(7) + 1 = 169 = 13^2$. Therefore, (8, 7, 13) is a solution.
(Note: the only other solution where x, y, and A are all less than 30 is (10, 6, 28).)

(Note also that if we don't require x, y, and A to be less than 30, (105, 104, 181) also works. Are there infinitely many solutions?)

2. Let R denote the number of red flowers and N the total number of flowers in the garden. The probability of picking 1 red flower when we pick a bunch of three flowers is

$$P(N) = \frac{\binom{R}{1}\binom{N-R}{2}}{\binom{N}{3}} = \frac{3R(N-R)(N-1-R)}{N(N-1)(N-2)}$$

Since 1/3 of all flowers are red we have that $R = \frac{N}{3}$. Substituting and simplifying,

$$P(N) = \frac{2N(2N-3)}{9(N-1)(N-2)}$$

Since we are told that the probability of picking exactly one red flower in a bunch of three is greater than 1/2,

$$\frac{2N(2N-3)}{9(N-1)(N-2)} > \frac{1}{2}$$

Simplifying, we obtain $N^2 - 15N + 18 < 0$

Using the quadratic formula we find N is between $\frac{15 - 3\sqrt{17}}{2}$ and $\frac{15 + 3\sqrt{17}}{2}$.
Since N is an integer and also a multiple of three, the largest possible $N = 12$.

3. If $x < 0$, then the left side of the equation $px^2 - qx + p = 0$ is positive. Therefore, If the equation has real solutions, then $x > 0$. Let $x = \frac{m}{n}$ where m and n are positive integers and $\frac{m}{n}$ is in lowest terms (i.e. m and n have no common factors other than 1). Substituting $x = \frac{m}{n}$ into the equation, and multiplying by n^2 to clear fractions, we obtain

$$pm^2 - qmn + pn^2 = 0 \text{ from which } pn^2 = qmn - pm^2 = m(qn - pm).$$

Therefore, m divides pn^2. But since m and n have no common factors, m must divide p. Similarly n must divide p as well. Since p is prime, $x = \frac{m}{n} = 1,\ p$, or $\frac{1}{p}$.

If $x = 1$, then $q = 2p$, which is impossible since q is prime. If $x = \frac{1}{p}$ or p, then the quadratic equation yields $q = p^2 + 1$. If p is odd, then q is an even number ≥ 10, again a contradiction since q is prime. Therefore, p must be even, so $p = 2$ and $q = 5$, and the only possible ordered pair $(p, q) = (2, 5)$. Thus, $P(x)$ is $2x^2 - 5x + 2$, and the roots of $P(x) = 0$ are 2 and $\frac{1}{2}$.

4. Let $x > 1$ be a real number satisfying $\langle x \rangle + \left\langle \frac{1}{x} \right\rangle = 1$. If n denotes the greatest integer in x, then $x = n + \langle x \rangle$, and obviously $n \geq 1$. Since $0 < \frac{1}{x} < 1$, we have $\left\langle \frac{1}{x} \right\rangle = \frac{1}{x}$.

Thus $x + \frac{1}{x} = n + \langle x \rangle + \left\langle \frac{1}{x} \right\rangle = n + 1$. Therefore, x is a solution to the quadratic equation $x^2 - (n+1)x + 1 = 0$. Using the quadratic formula, we find $x = \dfrac{(n+1) \pm \sqrt{(n+1)^2 - 4}}{2}$.

Since $x > 1$, we can eliminate the minus sign, so that $x = \dfrac{(n+1) + \sqrt{(n+1)^2 - 4}}{2}$.

If $n = 1$, then $x = 1$, which is a contradiction. Thus $n \geq 2$, and the potentially smallest x occurs when $n = 2$ and $x = \frac{3+\sqrt{5}}{2}$. To see that this number actually satisfies the given condition, note that $x \approx 2.6$, so $\langle x \rangle = x - 2 = \frac{-1+\sqrt{5}}{2}$. Also $\left\langle \frac{1}{x} \right\rangle = \frac{1}{x} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$.

Therefore, $\langle x \rangle + \left\langle \frac{1}{x} \right\rangle = \frac{-1+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2} = 1$. Thus the smallest value of x is $\frac{3+\sqrt{5}}{2}$.

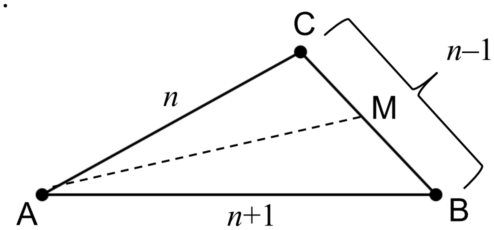
5. There are only three such ordered triples: (2, 3, 4), (3, 4, 5), and (7, 8, 9).

If ΔABC has sides of length 2, 3, and 4, then clearly the median to the side of length 4 creates an isosceles triangle with legs of length 2.

If ΔABC has sides of length 3, 4, and 5, it is a right triangle. The median to the hypotenuse of a right triangle is always half the length of the hypotenuse. Therefore, the median to the hypotenuse divides the triangle into two isosceles triangles with legs of length 2½.

Now represent the side lengths of ΔABC by $n-1$, n, and $n+1$.

Case 1: The median is drawn to the shortest side.



Since AM < AB, the only way either ΔAMC or ΔAMB could be isosceles is if AM = n or AM = $\frac{n-1}{2}$.

Using the law of cosines on ΔABC,

$$(n+1)^2 = n^2 + (n-1)^2 - 2n(n-1)\cos C \text{ from which } \cos C = \frac{n-4}{2n-2}.$$

Using the law of cosines on ΔAMC,

$$\text{AM}^2 = n^2 + \frac{(n-1)^2}{4} - 2(n)\left(\frac{n-1}{2}\right)\cos C$$

Suppose AM = n. Then

$$n^2 = n^2 + \frac{(n-1)^2}{4} - (n)(n-1)\cos C \text{ from which } \cos C = \frac{n-1}{4n}.$$

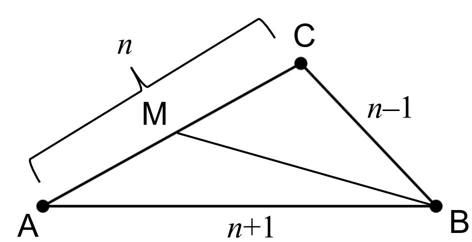
Therefore, $\frac{n-1}{4n} = \frac{n-4}{2n-2} \Rightarrow n^2 - 2n - 1 = 0$ which has no integer solutions.

Suppose AM = $\frac{n-1}{2}$. Again using the law of cosines on ΔAMC,

$$\left(\frac{n-1}{2}\right)^2 = n^2 + \frac{(n-1)^2}{4} - 2(n)\left(\frac{n-1}{2}\right)\cos C \text{ from which } \cos C = \frac{n}{n-1}.$$

Therefore, $\frac{n}{n-1} = \frac{n-4}{2n-2}$, whose only solution is $n = -4$. Thus the median cannot be to the shortest side.

Case 2: The median is drawn to the side of length n.



Since BM < AB, the only way either ΔAMC or ΔAMB could be isosceles is if BM = $n-1$ or BM = $\frac{n}{2}$.

From case 1, we know that $\cos C = \dfrac{n-4}{2n-2}$.

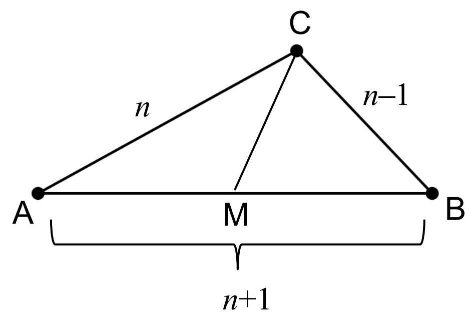
Using the law of cosines on ΔBMC, $BM^2 = \left(\dfrac{n}{2}\right)^2 + (n-1)^2 - 2\left(\dfrac{n}{2}\right)(n-1)\cos C$,

Substituting $\cos C = \dfrac{n-4}{2n-2}$, we obtain

$$BM^2 = \left(\frac{n}{2}\right)^2 + (n-1)^2 - 2\left(\frac{n}{2}\right)(n-1)\left(\frac{n-4}{2n-2}\right) \text{ from which } BM^2 = \frac{3n^2+4}{4}$$

If $BM = \dfrac{n}{2}$, then $\left(\dfrac{n}{2}\right)^2 = \dfrac{3n^2+4}{4}$, which has no real solutions.

If BM = n – 1, then $(n-1)^2 = \dfrac{3n^2+4}{4}$ from which $n^2 - 8n = 0$ and $n = 8$ yielding a triangle with sides of length 7, 8, and 9.



Case 3: The median is drawn to the longest side.

If AM = MB = CB, then $\dfrac{n+1}{2} = n-1$ from which $n = 2$. This gives a 2, 3, 4 triangle, which we have already considered.

If CM = MB = AM , then triangle ABC is a right triangle and the only right triangle with side lengths that are consecutive integers is the 3, 4, 5 triangle already considered.

The only other possibilities for ΔAMC or ΔAMB to be isosceles is if CM = n or n–1. Using the law of cosines on ΔABC,

$$(n-1)^2 = n^2 + (n+1)^2 - 2n(n+1)\cos A \text{ from which } \cos A = \frac{n+4}{2n+2}.$$

Using the law of cosines on ΔAMC,

$$CM^2 = n^2 + \left(\frac{n+1}{2}\right)^2 - 2n\left(\frac{n+1}{2}\right)\cos A = n^2 + \left(\frac{n+1}{2}\right)^2 - 2n\left(\frac{n+1}{2}\right)\left(\frac{n+4}{2n+2}\right) =$$

$$n^2 + \left(\frac{n+1}{2}\right)^2 - \frac{n(n+4)}{2}$$

If CM = n, then $n^2 = n^2 + \left(\dfrac{n+1}{2}\right)^2 - \dfrac{n(n+4)}{2}$. This becomes $n^2 + 6n - 1 = 0$, which has no integer solutions.

If CM = n – 1 , then $(n-1)^2 = n^2 + \left(\dfrac{n+1}{2}\right)^2 - \dfrac{n(n+4)}{2}$. This becomes $n^2 - 2n + 3 = 0$, which has no real solutions.

Therefore, the only triangles with the given properties have sides of lengths (2, 3, 4), (3, 4, 5), and

(7, 8, 9). The three triangles are shown below.