THE 2017-2018 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

Calculators are NOT permitted

Time allowed: 2 hours

1. The graph of the function $\dfrac{3^{x-y}}{2^{x+y}} = 8$ passes through only three of the four quadrants. Prove that the function is linear and identify, with proof, the quadrant through which the graph does not pass.

2. Ann and Ben are two very busy lawyers. They wish to have a meeting and agree to appear at a designated place on a certain day, but no earlier than noon and no later than 12:15 p.m. If necessary, Ann will wait 6 minutes for Ben to arrive, while Ben will wait 9 minutes for Ann to arrive, but neither can stay past 12:15 p.m. Compute, with proof, the probability that they will meet.

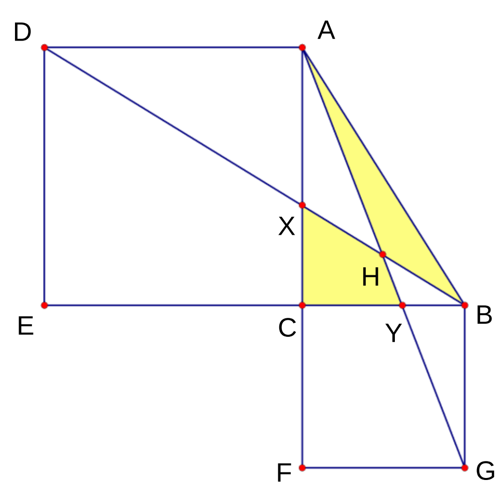
3. The coefficients a, b, and c of the equation $ax^2 + bx + c = 0$ are odd integers. Prove that there exists no ordered triple (a, b, c) for which the roots of the equation are rational.

4. A set of three or more distinct prime numbers is called *amazing* if the sum of every three of them is also a prime number. For example, the set {11, 23, 37, 79} is an amazing set of primes since 11 + 23 + 37 = 71 is prime, 11 + 23 + 79 = 113 is prime, 11 + 37 + 79 = 127 is prime, and 23 + 37 + 79 = 139 is prime. However, the set {5, 7, 11, 13} is not amazing since 5 + 7 + 13 = 25.

   a) Prove that no amazing set of four primes can contain the number 3.

   b) Prove that no amazing set of five primes exists.

5. In the figure at the right, ΔABC is a right triangle with right angle at C, and quadrilaterals ADEC and BCFG are squares. Point H is the intersection of $\overline{DB}$ and $\overline{AG}$. Prove that the area of quadrilateral CYHX is equal to the area of triangle AHB.



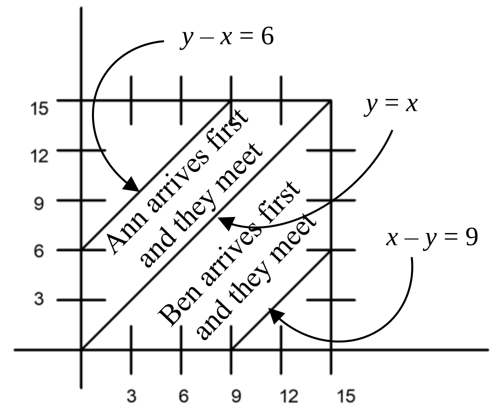
Solutions

1. Let $u = x - y$ and $v = x + y$. Then the equation becomes $\dfrac{3^u}{2^v} = 8 \;\Rightarrow\; 3^u = 8(2^v)$.

   Taking the log of both sides of the equation and applying well-known properties of logs, $u\log 3 = \log 8 + v\log 2$. Thus, $u = \dfrac{\log 8}{\log 3} + \dfrac{\log 2}{\log 3}v$. Therefore, $x - y = \dfrac{\log 8}{\log 3} + \dfrac{\log 2}{\log 3}(x + y)$.

   Rearranging terms and simplifying, we obtain $y = \dfrac{\log 3 - \log 2}{\log 3 + \log 2}x - \dfrac{\log 8}{\log 3 + \log 2}$. Clearly, this is linear with slope $\dfrac{\log 3 - \log 2}{\log 3 + \log 2}$ and y-intercept $\dfrac{-\log 8}{\log 3 + \log 2}$. Since the slope is positive and the y-intercept is negative, any point (x, y) on the function with a negative x-coordinate must have a negative y-coordinate. Therefore, the line never passes through the second quadrant.

2. Suppose Ann arrives x minutes past noon, and Ben arrives y minutes past noon. Then $0 \le x \le 15$, and $0 \le y \le 15$. If Ann arrives first, or they both arrive simultaneously, then $y \ge x$ and $0 \le y - x \le 6$ if the two are to meet. Similarly, if Ben arrives first, then $x \ge y$ and $0 \le x - y \le 9$ if the two are to meet.



   The sample space can be represented by a 15x15 square, and the probability that either person arrives first and they meet can be represented as the ratio of the area of each trapezoidal section shown to the area of the whole square. Since these two trapezoidal sections are mutually exclusive, the probability that Ann and Ben meet can be represented as the sum of these two ratios. Therefore, the desired probability is $\dfrac{225 - \frac{1}{2}(81) - \frac{1}{2}(36)}{225} = \dfrac{37}{50} = 74\%$.

3. Suppose $ax^2 + bx + c = 0$ has rational solutions for some ordered triple of odd integers (a, b, c).

   For the roots to be rational, $b^2 - 4ac$ must be the square of an integer. Let $b^2 - 4ac = k^2$. Then $b^2 - k^2 = (b - k)(b + k) = 4ac$. Since the right side of this equation is even and b is odd, k must also be odd. Let $b = 2m + 1$ and $k = 2n + 1$. Therefore,

   $$(b - k)(b + k) = (2m + 1 - 2n - 1)(2m + 1 + 2n + 1) = 4ac \quad \text{or} \quad (m - n)(m + n + 1) = ac.$$

   The product ac is odd. If m and n are each even, or if m and n are each odd, then $m - n$ is even, which is not possible. If one is even and the other odd, then $(m + n + 1)$ is even, which is also impossible for all possibilities. Hence, for all possibilities, $(m - n)(m + n + 1)$ is even and we have a contradiction. Therefore, no such ordered triple (a, b, c) of odd integers exists.

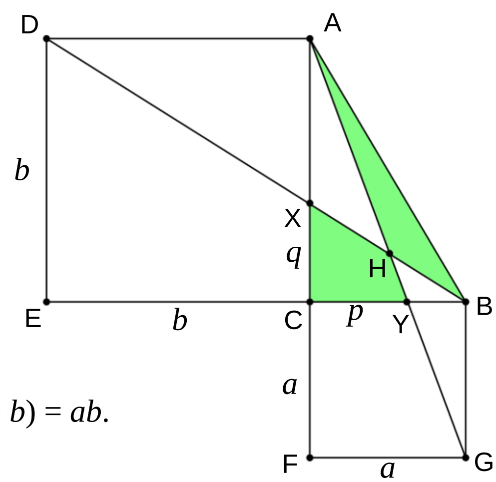
4. a) Any integer n can be written in the form $n = 3k + r$ where k is an integer and $r = 0, 1,$ or 2. Let us refer to these as type r, where $r = 0, 1,$ or 2. The only prime number for which $r = 0$ is 3 itself. Suppose S is an amazing set of four primes, one of which is 3. Represent the three remaining primes as p_1 , p_2 , and p_3 .

Claim: These three primes cannot all be of the same type r. Suppose they were. Then $p_1 + p_2 + p_3 = 3(k_1 + k_2 + k_3) + 3r$ which is divisible by 3. Thus, $p_1 + p_2 + p_3 > 3$ and is not prime, contradicting the fact that S is amazing.

Therefore, there exist primes p_1 and p_2 in S such that $p_1 = 3k_1 + 1$ and $p_2 = 3k_2 + 2$. Since $3 + p_1 + p_2 = 3 + 3k_1 + 1 + 3k_2 + 2 = 3(k_1 + k_2 + 2)$, their sum is not prime, contradicting the fact that S is amazing. Thus 3 cannot be a member of any amazing set S with 4 or more elements.

b) Suppose S has 5 elements. Then from part (a), there can be at most two of the same type. Since 3 cannot be an element of S, at least one of the other two types must occur three times, which is a contradiction. Therefore, no amazing set can have more than four elements.

5. Represent the length of the side of square CBGF as a and the length of the side of square ADEC as b. Let CY = p and CX = q.



Since $\overline{CY}$ is parallel to $\overline{FG}$, $\Delta ACY \sim \Delta AFG$.

Therefore, $\dfrac{p}{a} = \dfrac{b}{a+b} \Rightarrow p(a + b) = ab$.

Similarly, $\Delta DEB \sim \Delta XCB$. Therefore, $\dfrac{b}{q} = \dfrac{a+b}{a} \Rightarrow q(a + b) = ab$.

Thus $p = q$.

Since YB = $a - p$, the area of $\Delta AYB = \frac{1}{2}b(a-p)$. The area of $\Delta XBC = \frac{1}{2}aq$.

The area of ΔABH = area ΔAYB – area ΔHYB = $\frac{1}{2}b(a-p)$ – area ΔHYB.

The area of quad XCYH = area ΔXBC – area ΔHYB = $\frac{1}{2}aq$ – area ΔHYB.

From $p(a + b) = ab$ we obtain $pa = ab - pb = b(a - p)$.

Therefore, area of $\Delta ABH = \frac{1}{2}b(a-p)$ – area ΔHYB = $\frac{1}{2}pa$ – area ΔHYB. Since $p = q$, this is also the area of quad XCYH.