Kennesaw State University
College of Science and Mathematics
Department of Mathematics

THE 2018-2019 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

Calculators are <u>NOT</u> permitted

Time allowed: 2 hours

1. Let m be a three-digit integer with distinct digits. Find all such integers m which are equal to the average (arithmetic mean) of the six numbers obtained by forming all possible arrangements of the digits of m. Prove that you have found them all.

2. A bag contains N balls, some of which are red and the rest yellow. Two balls are drawn randomly from the bag, without replacement. If the probability that the two balls are the same color is equal to the probability that they are different colors, compute, with proof, the set of all possible values of N.

3. Let $P(x)$ be a polynomial with integer coefficients. Suppose that $P(0)$ is an odd integer and that $P(1)$ is also an odd integer. Prove that $P(c)$ is an odd integer for all integers c.

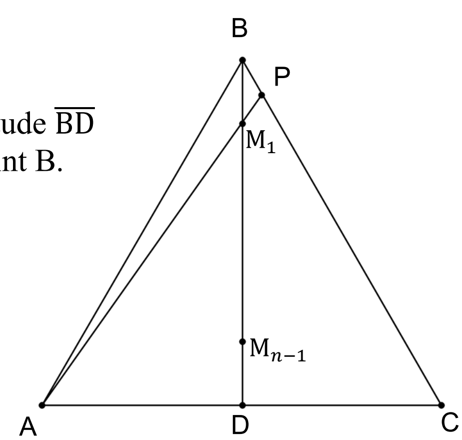
4. Let S = a set of consecutive positive integers beginning with 1. All subsets of S that do <u>not</u> contain two consecutive numbers are formed. The product of the elements in each subset is calculated. (Note: if a subset contains only one number, the product of its elements is the number itself.) Let N = the sum of the squares of all these products.

   For example, if $S = \{1, 2, 3\}$, then the allowable subsets are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$. The products are 1, 2, 3, and 3, and $N = 1^2 + 2^2 + 3^2 + 3^2 = 23$.

   Find, with proof, the value of N when $S = \{1, 2, 3, \ldots, 16\}$.

5. In equilateral ΔABC, points $M_1, M_2, M_3, \ldots M_{n-1}$ divide altitude $\overline{BD}$ into n segments of equal length ($n > 1$), with M_1 closest to point B. Segment $\overline{AM_1}$ is extended to meet side $\overline{BC}$ at point P.

   Prove that $\dfrac{AM_1}{M_1P} = 2n - 1$.



Solutions

1. Suppose $m = 100a + 10b + c$. If we arrange the digits of m in all six possible ways, then each of a, b, and c will occur exactly twice in the 1's place, twice in the 10's place, and twice in the 100's place. Thus, the sum of the six arrangements is

$$2(a + b + c)\,(100 + 10 + 1) = 222(a + b + c).$$

Hence, the arithmetic mean of these six numbers is $37(a + b + c)$.

Since we are given $100a + 10b + c = 37(a + b + c)$, we have $7a = 3b + 4c$.

Method 1: $7a = 3b + 4c \;\Rightarrow\; 3b + 4c$ is a multiple of 7. Make a chart, remembering that all three digits are distinct. The values of c in the middle column of the chart are the only ones for which $3b + 4c$ is a multiple of 7.

b	$3b + 4c$	c	$7a = 3b + 4c$	a
0	$4c$	7	28	4
1	$4c + 3$	8	35	5
2	$4c + 6$	9	42	6
3	$4c + 9$	none		
4	$4c + 12$	none		
5	$4c + 15$	none		
6	$4c + 24$	none		
7	$4c + 21$	0	21	3
8	$4c + 24$	1	28	4
9	$4c + 27$	2	35	5

Therefore, there are a total of six values of m: 370, 407, 481, 518, 592, and 629.

Method 2: $7a = 3b + 4c \;\Rightarrow\; 7(a – c) = 3(b – c)$.

Now, $-9 \le b – c \le 9$ and 7 divides $b – c$. There are just three possibilities.

If $b – c = 0$, then $a = b = c$, which cannot be since the digits of m are all different.

If $b – c = 7$, then $a – c = 3$. In this case $b = c + 7$ and $a = c + 3$, leaving only three possibilities for c, namely 0, 1, and 2. These yield m = 370, 481, and 592.

Finally, if $b – c = –7$, then $a – c = –3$. In this case $b = c – 7$ and $a = c – 3$, again leaving only three possibilities for c, namely 7, 8, and 9. These yield m = 407, 518, and 629.

Therefore, there are a total of six values of m: 370, 407, 481, 518, 592, and 629.

2. Let K = the number of red balls and $N - K$ the number of yellow balls. Then, the probability of drawing two balls of the same color is $\left(\frac{K}{N}\right)\left(\frac{K-1}{N-1}\right)+\left(\frac{N-K}{N}\right)\left(\frac{N-K-1}{N-1}\right)$. Since the probability of drawing two balls of the same color is equal to the probability of drawing two balls of different colors, each probability is one-half. Therefore,

$$\left(\frac{K}{N}\right)\left(\frac{K-1}{N-1}\right)+\left(\frac{N-K}{N}\right)\left(\frac{N-K-1}{N-1}\right)=\frac{1}{2}$$

Simplifying, $4K^2 - 4NK + N(N-1) = 0$.

Using the quadratic formula, the roots of this equation are $K = \frac{N+\sqrt{N}}{2}$ and $K = \frac{N-\sqrt{N}}{2}$.

Noting that N and $\sqrt{N}$ have the same parity if N is a perfect square, then K will be an integer if and only if N is a perfect square. The required condition is satisfied for all values of N which are squares of integers and no others. (Note: If $N = 1$, the probability of picking two balls of the same or of different colors is zero, which still works.)

3. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

First, we prove that the number of odd integers in the set $\{a_n,\ a_{n-1}, \dots, a_1\}$ is even.

Substituting, $P(0) = a_0$ and $P(1) = a_n + a_{n-1} + \cdots + a_1 + a_0$.
Thus, by the given information, a_0 is odd and $a_n + a_{n-1} + \cdots + a_1 + a_0$ is odd.

Since the difference between two odd integers is always even, $a_n + a_{n-1} + \cdots + a_1$ is even.
This can only happen if the number of odd integers in the set $\{a_n, a_{n-1}, \dots, a_1\}$ is even.

Consider $P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$.

Suppose c is even. Then $a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c = c(a_n c^{n-1} + a_{n-1}c^{n-2} \dots + a_1)$ is even. Adding a_0, which is odd, makes $P(c)$ odd.

Suppose c is odd. Since $\{a_n,\ a_{n-1}, \dots, a_1\}$ contains an even number of odd integers, $\{a_n c^n, a_{n-1}c^{n-1}, \dots, a_1 c\}$ contains an even number of products of odd integers, and therefore, must have an even sum.

Therefore, $a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c$ is even and since a_0 is odd,
$P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$ must be odd.

Thus, $P(c)$ is odd for all integers c.

4. To get an idea, we first look at simpler versions of the problem.

Consider the set $\{1, 2\}$. The allowable subsets are $\{1\}$, $\{2\}$. The products are 1, 2, and the sum of their squares is 5.

For the set $\{1, 2, 3\}$, we already know the sum of the squares is 23.

Consider the set $\{1, 2, 3, 4\}$. The allowable subsets are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{2, 4\}$ and $\{1, 4\}$. The products are 1, 2, 3, 4, 3, 8, 4, and the sum of their squares is 119.

Consider the set $\{1, 2, 3, 4, 5\}$. The allowable subsets are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 5\}$ and $\{1, 3, 5\}$. The products are 1, 2, 3, 4, 5, 3, 4, 5, 8, 10, 15, and 15, and the sum of their squares is 719.

Observe that the sum of the squares of the products appears to be $(n + 1)! - 1$, where n is the number of elements in the set S. We prove that this expression is correct by mathematical induction.

For $n = 1$, the formula is trivially true.

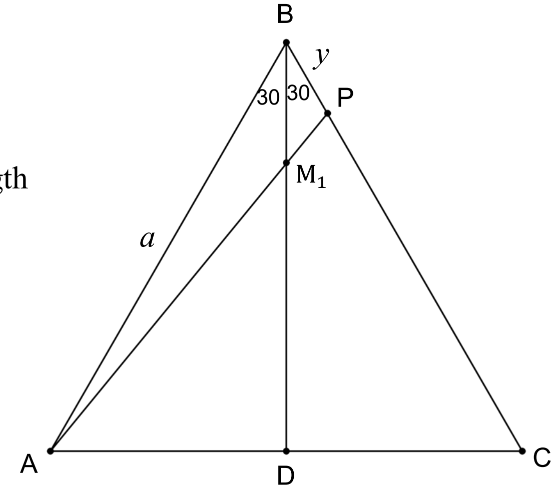
Assume that for a set consisting of the first K positive integers, the desired result is $(K + 1)! - 1$.

Consider the set consisting of the first $(K + 1)$ positive integers. Split the collection of subsets with no consecutive numbers into two subcollections: subsets containing $K + 1$, and those that don't.

Each subset in the first subcollection can be represented as the union of $\{K + 1\}$ and the non-empty subsets of $\{1, 2, 3, \ldots, K-1\}$ which do not contain consecutive numbers. Therefore, by the induction hypothesis, the sum of the squares of the products from this first subcollection is $(K+1)^2(K!-1)+(K+1)^2$. Similarly, the sum of the squares of the products from the second subcollection is $(K + 1)! - 1$.

Adding $(K+1)^2(K!-1)+(K+1)^2$ and $(K + 1)! - 1$ and simplifying, we obtain $(K + 2)! - 1$. Therefore, the desired value is $17! - 1$ (which, by the way, is equal to 333,688,130,096,000).

5. Method 1



Let AB = BC = a and BP = y

Then the length of altitude BD is $\frac{a\sqrt{3}}{2}$ and the length of $\overline{BM_1}$ is $\frac{a\sqrt{3}}{2n}$

Since BM_1 is an angle bisector in ΔABP,

$$\frac{AM_1}{M_1P} = \frac{AB}{BP} = \frac{a}{y}.$$

The area of $\Delta ABM_1 = \frac{1}{2}(AB)(BM_1)\sin 30 = \frac{1}{2}(a)BM_1\left(\frac{1}{2}\right) = \frac{1}{4}(a)BM_1$.

Similarly, the area of $\Delta BM_1P = \frac{1}{2}(BP)(BM_1)\sin 30 = \frac{1}{2}(y)BM_1\left(\frac{1}{2}\right) = \frac{1}{4}(y)BM_1$

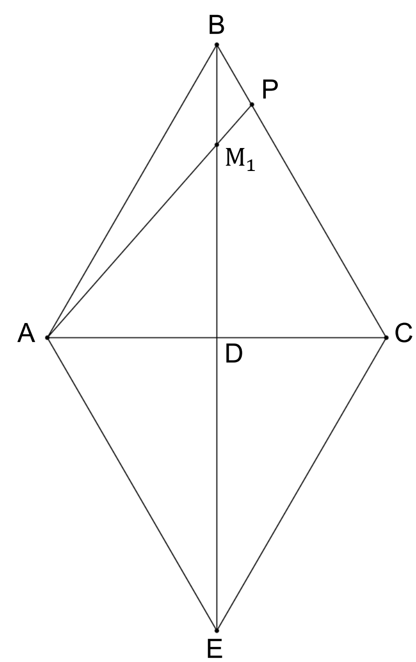
and the area of $\Delta ABP = = \frac{1}{2}(AB)(BP)\sin 60 = \frac{1}{2}ay\frac{\sqrt{3}}{2} = \frac{1}{4}ay\sqrt{3}$.

Therefore, since area ΔABM_1 + area ΔBM_1P = area ΔABP,

$$\frac{1}{4}aBM_1 + \frac{1}{4}yBM_1 = \frac{1}{4}ay\sqrt{3} \Rightarrow BM_1\,(a+y) = ay\sqrt{3} \Rightarrow \frac{a\sqrt{3}}{2n}(a+y) = ay\sqrt{3}$$

Thus, $\frac{a+y}{2n} = y$, from which $a = 2ny - y \Rightarrow \frac{a}{y} = 2n - 1 = \frac{AM_1}{M_1P}$.

Method 2 (from Russell Emerine – Walton High School)



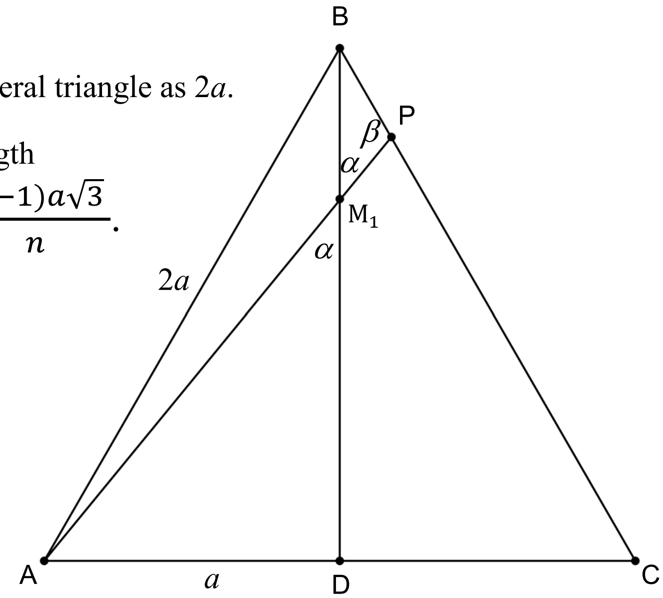
Reflect ΔABC over side AC and label the image of B point E.

Let $BM_1 = x$. Then $M_1D = (n-1)x$ and $DE = nx$ by symmetry,
So that $M_1E = (2n-1)x$.

Since $m\angle AEM_1 = m\angle PBM_1 = 30$, and $\angle BM_1P$ and $\angle AM_1E$ Are congruent vertical angles, $\Delta BM_1P \sim \Delta EM_1A$.

Therefore, $\frac{AM_1}{M_1P} = \frac{EM_1}{BM_1} = \frac{(2n-1)x}{x} = 2n - 1$.

Method 3



Represent the length of the sides of the equilateral triangle as $2a$.

Then the length of altitude BD is $a\sqrt{3}$, the length of $\overline{BM_1}$ is $\dfrac{a\sqrt{3}}{n}$, and the length of $\overline{M_1D}$ is $\dfrac{(n-1)a\sqrt{3}}{n}$.

Using the Pythagorean Theorem on ΔADM_1,

$AM_1 = \dfrac{a\sqrt{4n^2-6n+3}}{n}$.

Let $m\angle AM_1D = m\angle BM_1P = \alpha$.

From right triangle ADM_1,

$\sin\alpha = \dfrac{AD}{AM_1} = \dfrac{n}{\sqrt{4n^2-6n+3}}$ and

$\cos\alpha = \dfrac{M_1D}{AM_1} = \dfrac{(n-1)\sqrt{3}}{\sqrt{4n^2-6n+3}}$.

Let $m\angle BPM_1 = \beta$. Noting that $m\angle M_1BP = 30$, then $\beta = 180 - (30 + \alpha)$.

Hence, $\sin\beta = \sin[180 - (30+\alpha)] = \sin(30+\alpha) = (\sin 30)(\cos\alpha) + (\cos 30)(\sin\alpha)$.

Thus, $\sin\beta = \dfrac{1}{2}\dfrac{(n-1)\sqrt{3}}{\sqrt{4n^2-6n+3}} + \dfrac{\sqrt{3}}{2}\dfrac{n}{\sqrt{4n^2-6n+3}} = \dfrac{\sqrt{3}(2n-1)}{2\sqrt{4n^2-6n+3}}$

Using the Law of Sines on ΔBPM_1, $\dfrac{\sin 30}{M_1P} = \dfrac{\sin\beta}{BM_1} \Rightarrow \dfrac{\frac{1}{2}}{M_1P} = \dfrac{\frac{\sqrt{3}(2n-1)}{2\sqrt{4n^2-6n+3}}}{\frac{a\sqrt{3}}{n}}$.

Hence, $M_1P = \dfrac{a\sqrt{4n^2-6n+3}}{n(2n-1)}$. Finally, $\dfrac{AM_1}{M_1P} = \dfrac{\frac{a\sqrt{4n^2-6n+3}}{n}}{\frac{a\sqrt{4n^2-6n+3}}{n(2n-1)}} = 2n - 1$.