THE 2019–2020 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

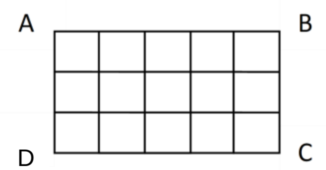
PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. Rectangle ABCD is made up of 15 congruent squares. How many rectangles with length equal to twice the width are contained in the given figure?

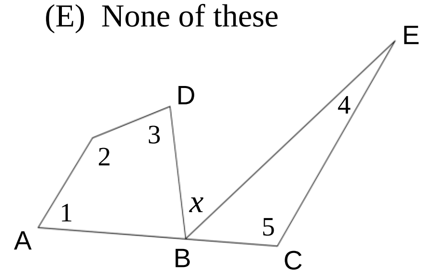


(A) 26 (B) 24 (C) 22 (D) 18 (E) 12

2. Consider the following two sets of positive integers: A = {numbers which are 2019 more than a positive prime}, and B = {numbers which are 2019 times a positive prime}. How many numbers are in $A \cap B$, the intersection of sets A and B?

(A) 1 (B) 2 (C) 3 (D) 4 (E) None of these

3. In the diagram, points A, B, and C are collinear. Express the sum of the measures of the five numbered angles as a function of x, the measure of angle DBE.



(A) $180 + x$ (B) $270 + x$ (C) $360 - x$ (D) $360 + x$ (E) $450 - x$

4. If $6^{2-3y} = 2$, compute the value of 6^{2+3y}.

(A) 108 (B) 216 (C) 324 (D) 648 (E) None of these

5. Parents representing 26 different families met together at the annual meeting of MBA (Multiple Births Association). Each of the 26 families had exactly one set of either twins, triplets, or quadruplets. To everyone's surprise, the number of children that were part of twins, the number of children that were part of triplets, and the number of children that were part of quadruplets were all the same. How many of the 26 families had triplets?

(A) 9 (B) 8 (C) 7 (D) 6 (E) 5

6. Suppose that a and b are positive integers, c is a real number, and i is the imaginary unit. If $(a + bi)^3 = -74 + ci$, compute $a - b$.

(A) –5 (B) –4 (C) –3 (D) –2 (E) –1

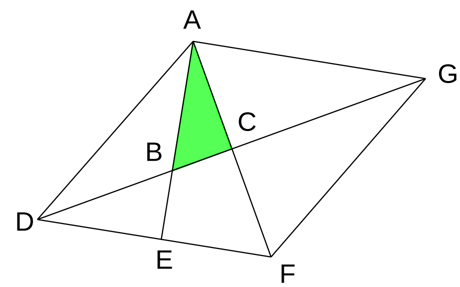
7. Debbie and Don both made the same trip of 220 miles. Don made the trip in 20 minutes less time than Debbie, averaging 5 mph faster than Debbie. What was Don's average speed?

(A) 50 (B) 55 (C) 60 (D) 65 (E) None of these

8. Determine the number of four-digit positive integers which are divisible by 12 and whose digits are all primes.

(A) 16 (B) 15 (C) 14 (D) 13 (E) None of these

9. In the diagram, ADFG is a rhombus and $\overline{AE}$ is an altitude of the rhombus. How many triangles in the diagram must be similar to ΔABC (do <u>not</u> include ΔABC itself)?



(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

10. Dr. Garner has 20 students in his calculus class. When calculating the class average on his most recent test, he accidentally entered the digits of Abby's test score in the reverse order, and he obtained an average that was 2.7 less than it should have been. If Abby's actual score was greater than 60, and the maximum possible score was 100, what is the sum of all possible scores Abby could have received?

(A) 172 (B) 246 (C) 310 (D) 318 (E) None of these

11. If x is a positive acute angle such that log (sin x) + log (cos x) + log (tan x) = –1, compute the numerical value of $\cot x$.

(A) $\frac{1}{3}$ (B) $\frac{1}{\sqrt{3}}$ (C) 3 (D) $\sqrt{3}$ (E) $3\frac{1}{3}$

12. Two distinct numbers are selected at random from the set $\{1, 2, 3, 4, \ldots, 14, 15\}$.

What is the probability that the absolute value of the difference of the numbers is 5 or more?

(A) $\frac{1}{2}$ (B) $\frac{9}{21}$ (C) $\frac{11}{21}$ (D) $\frac{18}{35}$ (E) None of these

13. If 297_b is a factor of 792_b (where 297_b means 297 in base *b*). Compute the value of 297_b in base 10.

(A) 525 (B) 592 (C) 720 (D) 900 (E) 975

14. Let *f* be a real-valued function such that $f(x) + 3xf\left(\frac{1}{x}\right) = 2(x + 1)$ for all real numbers $x > 0$. Find the value of *f*(2019).

(A) 1010 (B) 1011 (C) 2019 (D) 2020 (E) None of these

15. The number 53 has an interesting property. When $53^2 = 2809$ is divided by 100, the remainder (9) is the square of the units digit of 53. How many integers from 10 to 99, inclusive, have the property that the remainder from dividing x^2 by 100 is equal to the square of the units digit of *x*?

(A) 24 (B) 26 (C) 28 (D) 30 (E) 32

16. The ratio of the measures of the interior angles of two regular polygons is 7:8. Let *n* and *N* represent the number of sides of two such polygons (with $n < N$). Compute the sum of all possible values for *n*.

(A) 43 (B) 44 (C) 50 (D) 58 (E) 60

17. Two parallel lines cross both the *x* and *y*-axes. The intercepts on the *x*-axis are 20 cm apart and the intercepts on the y-axis are 21 cm apart. If the lines are *k* units apart, which of the following is closest to *k*?

(A) 14.1 (B) 14.2 (C) 14.3 (D) 14.4 (E) 14.5

18. A magic square is composed of distinct positive integers such that the sum of the numbers in every row, every column, and the two main diagonals is the same. If all remaining numbers in the 3x3 magic square shown are either one-digit or two-digit prime numbers, compute the value of *n*.

67	1	
n		

(A) 7 (B) 17 (C) 19 (D) 23 (E) 31

19. Suppose that *x* and *y* are real numbers such that $x^2 \neq y^2$. If $x^3 = 17x + 5y$ and $y^3 = 5x + 17y$, compute the value of $(x^2 - y^2)^2$.

(A) 169 (B) 189 (C) 249 (D) 289 (E) None of these

20. The first term of an arithmetic sequence of distinct terms is 4. The 1st, 5th, 15th and kth terms of the arithmetic sequence form a geometric sequence in the same order. Compute the value of k.

(A) 25 (B) 35 (C) 36 (D) 39 (E) 40

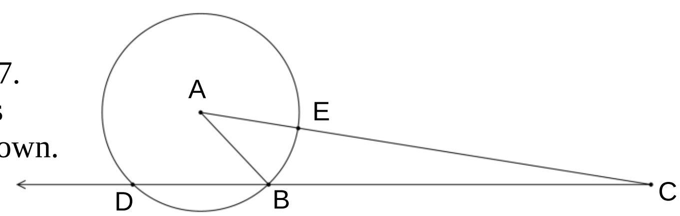
21. For which of the following values of x will $\frac{(x+10395)^2}{x}$ be an integer, while $\frac{x+10395}{x}$ is not?

(A) 2079 (B) 3575 (C) 5136 (D) 6237 (E) None of these

22. Let p, q, and r be the roots of the equation $x^3 - 14x^2 + 29x - 4 = 0$. Compute the value of the expression $\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)$.

(A) $-\frac{7}{2}$ (B) $\frac{1}{4}$ (C) -3 (D) -12 (E) $\frac{29}{2}$

23. In triangle ABC, AB = 7, BC = 33, and AC = 37. A circle centered at A with radius AB intersects ray CB at point D and side AC at point E, as shown. Compute the distance from D to E.



(A) $\frac{70}{\sqrt{37}}$ (B) $\frac{7\sqrt{33}}{3}$ (C) $\frac{10\sqrt{37}}{\sqrt{33}}$ (D) $\frac{7\sqrt{37}}{3}$ (E) $\frac{70}{\sqrt{33}}$

24. The sum of the integers from 1 to 48, inclusive, is exactly six times the sum of the integers from A to B, inclusive, where A and B are both integers and $1 < A < B < 48$. How many such ordered pairs (A, B) are there?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

25. In the country of Newtonia, the basic unit of money used to be the newton. However, newtons are no longer being made. They have been replaced with forinths and schillings. A forinth is worth m newtons and a schilling is worth n newtons, m and n integers, $m < n$. Schillings and forinths can be combined to make all but 35 monetary values of newtons. In particular, 58 newtons cannot be made from schillings and forinths. Compute the ordered pair (m, n).

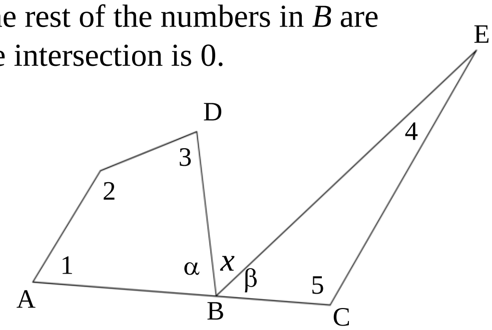
(A) (6, 15) (B) (8, 11) (C) (10, 13) (D) (11, 14) (E) None of these

Solutions

1. A Each of the three rows has four 1x2 rectangles. Each of the five columns has two 1x2 rectangles. There are a total of four 2x4 rectangles. The total is 26.

2 . E Since 2 is the only even prime number, the smallest number in set *A* is 2 + 2019 = 2021. All the rest of the numbers in *A* are the sum of two odd numbers and are, therefore, even. The smallest number in set *B* is (2)(2019) = 4038, and all the rest of the numbers in *B* are odd. Thus $A \cap B$ is empty and the number of elements in the intersection is 0.

3. D $1 + 2 + 3 + \alpha = 360$ and $4 + 5 + \beta = 180$.
Adding these two equations:
$1 + 2 + 3 + 4 + 5 + \alpha + \beta = 540$. Also $\alpha + \beta = 180 - x$.
Therefore, $1 + 2 + 3 + 4 + 5 + 180 - x = 540$ from which
$1 + 2 + 3 + 4 + 5 = 360 + x$.



4. D $6^{2-3y} = \frac{6^2}{6^{3y}} = \frac{36}{6^{3y}} = 2$. From this, $6^{3y} = 18$. Therefore, $6^{2+3y} = (6^2)(6^{3y}) = (36)(18) = 648$

5. B Let *T* = # of families with twins, *R* = # of families with triplets, and *Q* = # of families with quadruplets. Then we are given $T + R + Q = 26$ and $2T = 3R = 4Q$. From the second equation, $T = \frac{3}{2}R$ and $Q = \frac{3}{4}R$. Therefore, $\frac{3}{2}R + R + \frac{3}{4}R = 26$ and $R = 8$.

6. B When $(a + bi)^3$ is expanded, the real part is $a^3 - 3ab^2$. Therefore, $a^3 - 3ab^2 = a(a^2 - 3b^2) = -74$. Since *a* and *b* are positive integers, and *a* is a factor of 74, a little trial and error gives $a = 1$ and $b = 5$ as the only solution, and $a - b = -4$.

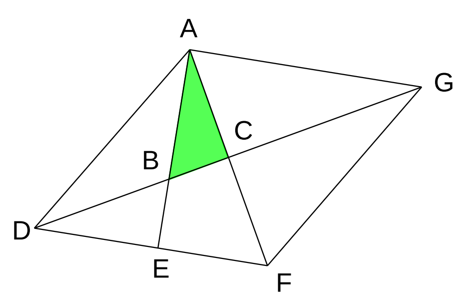
7. C Let r = Debbie's rate, $r + 5$ = Don's rate, $\frac{220}{r}$ = Debbie's time, $\frac{220}{r} - \frac{1}{3}$ = Don's time

Don's distance = Don's rate × Don's time. Thus, $220 = (r+5)\left(\frac{220}{r} - \frac{1}{3}\right) \Rightarrow r = 55$

Therefore, Debbie's average speed is 55 mph and Don's is 60 mph.

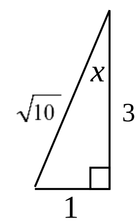
8. A The prime digits are 2, 3, 5, 7. Because 4 is a factor, the last two digits must be 32, 52 or 72. Because 3 is a factor, the sum of the digits must be a multiple of 3.
The possibilities are: 2232, 2352, 2532, 2772, 3252, 3372, 3552, 3732, 5232, 5352, 5532, 5772, 7272, 7332, 7572, 7752, for a total of 16.

9. D There are 7 triangles that are similar to ΔABC. They are:
ΔDBE, ΔAFE, ΔGAC, ΔGFC, ΔDFC, ΔDAC, and ΔGBA.



10. B Let the correct two-digit score be 10A + B. Then the miss-entered score was 10B + A. Since the class average was 2.7 points less than it should have been, and there are 20 students in the class, Abby's miss-entered score was (20)(2.7) = 54 points less than her actual score. Thus, 10A + B – 54 = 10B + A. From this, 9A – 9B = 54 or A – B = 6. Since Abby's actual score was greater than 60, the possibilities are 71, 82 or 93 and the desired sum is 246.

11. C Using properties of logs,
log (sin x) + log (cos x) + log (tan x) = log[(sin x)(cos x)(tan x)] = log(sin$^2 x$) = –1
Therefore, $\sin^2 x = 10^{-1} = \frac{1}{10}$ and, since x is a positive acute angle, $\sin x = \frac{1}{\sqrt{10}}$
Using the right triangle shown, cot x = 3.



12. C Consider the possibilities:

Smaller number	1	2	3	4	5	6	7	8	9	10
# of possible pairs	10	9	8	7	6	5	4	3	2	1

Thus the total number of pairs is 55. There are ${}_{15}C_2$= 105 ways of choosing two numbers, so the desired probability is $\frac{55}{105} = \frac{11}{21}$.

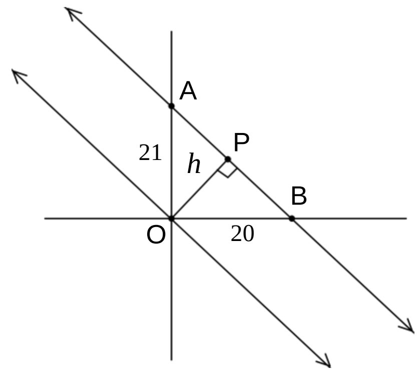
13. D We could set $2b^2 + 9b + 7$ equal to each of the choices and solve for b, but here is a less tedious approach.
Note that $b \geq 10$. We also know 2(297) < 2(300) = 600 < 792 < 800 = 4(200) < 4(297).
Therefore, $792_b = 3(297_b)$, which means $7b^2 + 9b + 2 = 3(2b^2 + 9b + 7)$. Solving gives b = 19 so that 297_b = 900.

14. A Since $f(x) + 3xf\left(\frac{1}{x}\right) = 2(x + 1)$, for all $x > 0$, then $f\left(\frac{1}{x}\right) + 3\left(\frac{1}{x}\right) f(x) = 2(\frac{1}{x} + 1)$ for all $x > 0$. Solving this system of equations for $f(x)$ and $f\left(\frac{1}{x}\right)$ gives $f(x) = \frac{x+1}{2}$.
Therefore, $f(2019)$ = 1010.

15. B Represent the two-digit number as $10t + u$. Then $(10t + u)^2 = 100t^2 + 20tu + u^2$. Thus, for the remainder when the number is divided by 100 to be equal to u^2, we need 100 to divide $20tu$, or equivalently 5 to divide t or u. Therefore, either t = 5 or u = 5 or u = 0. There are 10 numbers whose tens digit is 5 and 18 whose units digit is 5 or 0, but two of these (50 and 55) are counted twice. Hence, there are 10 + 18 – 2 = 26 such numbers.

16. D Using the formula for the measure of the interior angle of a regular polygon,
$\frac{7}{8} \cdot \frac{(N-2)180}{N} = \frac{(n-2)180}{n}$. Simplifying and solving for N, $N = \frac{14n}{16-n}$.
Noting that $n > 2$ and that n and N are positive integers, the only acceptable values of n are 8, 9, 12, 14, 15 and the desired sum is 58.

17. E Translate both lines so that one of them passes through the origin. Since translation preserves distance, the information can be represented as in the diagram shown. Then the distance between the lines is the length of the altitude of a right triangle with legs 20 cm and 21 cm long. Using the Pythagorean Theorem, the length of the hypotenuse of right triangle AOB is 29 cm.



Since ΔPOB is similar to ΔOBA, $\frac{h}{20} = \frac{21}{29}$ and $h = \frac{420}{29} \approx 14.48$.

Choice E is closest.

(Or we could use the area of ΔPOB: $\frac{1}{2}(20)(21) = \frac{1}{2}(29)(h)$).

18. E Let the numbers in the upper and lower right squares be a and b, as shown. The sum of the numbers in any row, column, or diagonal is $68 + a$. The missing numbers in the second row, from right to left, will be $(68 - b)$, $(1 + a - b)$, and $(2b - 1)$.

67	1	a
n		b

The missing numbers in the last row will be $n = (2 + a - 2b)$ and $(66 + b)$. The sum of the numbers in a diagonal must also be $68 + a$, therefore,

$$(2 + a - 2b) + (1 + a - b) + a = 68 + a \text{ or } b = \frac{2a-65}{3}.$$

Solving this equation for prime numbers a and b, the possibilities are: $(a, b) = (37, 3), (43, 7), (61, 19), (67, 23), (79, 31), (97, 43)$. Only $(43, 7)$ produces all primes in the remaining spaces, as shown. Thus, $n = (2 + a - 2b) = 31$.

67	1	43
13	37	61
31	73	7

19. B Adding the two equations gives $x^3 + y^3 = 22x + 22y = 22(x + y)$. Factoring the left and noting that $x + y \neq 0$, we obtain $x^2 - xy + y^2 = 22$.
Subtracting the two equations gives $x^3 - y^3 = 12x - 12y = 12(x - y)$.
Factoring the left and noting that $x - y \neq 0$, we obtain $x^2 + xy + y^2 = 12$.
Adding the two results gives $2x^2 + 2y^2 = 34$ or $x^2 + y^2 = 17$.
Substituting into $x^2 - xy + y^2 = 22$ we find $xy = -5$.
Now, $(x^2 - y^2)^2 = (x^2 - y^2)(x^2 - y^2) = (x - y)^2(x + y)^2 = (x^2 - 2xy + y^2)(x^2 + 2xy + y^2)$ $= (17 + 10)(17 - 10) = 189$.

20. E Let the terms of the arithmetic sequence be $4, 4 + d, 4 + 2d, \ldots$, with $d \neq 0$. Since the 1st, 5th, and 15th terms form a geometric sequence, $\frac{4+4d}{4} = \frac{4+14d}{4+4d} \Rightarrow d = \frac{3}{2}$. Therefore, the first three terms of the geometric sequence are 4, 10, 25, with a constant ratio of $\frac{5}{2}$, and the 4th term is $25\left(\frac{5}{2}\right) = \frac{125}{2}$. The kth term of the arithmetic sequence is $4 + \frac{3}{2}(k - 1)$.

Therefore, $4 + \frac{3}{2}(k - 1) = \frac{125}{2} \Rightarrow k = 40$.

21. D The problem could be done by direct calculation using each of the choices, but that is very tedious. Here is a shorter approach.

Simplifying the expressions $\frac{(x+y)^2}{x} = \frac{x^2+2xy+y^2}{x} = x+2y+\frac{y^2}{x}$, and $\frac{x+y}{x} = 1+\frac{y}{x}$.

Therefore, we are looking for a value of x that divides 10395^2, but not 10395. Factoring $10395 = (3^3)(5)(7)(11)$. Factoring each of the choices,

Choice A: $2079 = (3^3)(7)(11)$, which divides 10395, so it can be eliminated.

Choice B: $3575 = (5^2)(11)(13)$, which cannot divide 10395^2 so it can be eliminated.

Choice C: 5136 can be eliminated because it is even.

Choice D: $6237 = (3^4)(7)(11)$, has the right combination of factors.

22. C In any equation of the form $x^3+ax^2+bx+c=0$ with roots p, q, and r,

(i) $p+q+r = -a$, (ii) $pq+pr+qr = b$, and (iii) $pqr = -c$

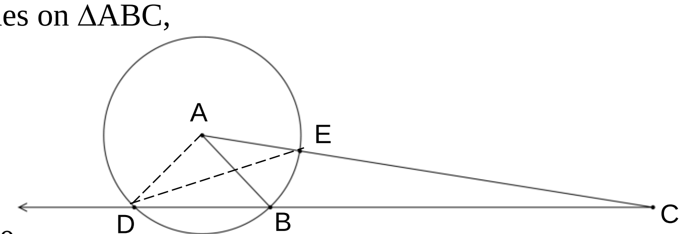
$$\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right) = \frac{(p-1)(q-1)(r-1)}{pqr} = \frac{(p-1)(q-1)(r-1)}{4} \quad \text{from (iii)}$$

Let $P(x) = x^3 - 14x^2 + 29x - 4 = (x-p)(x-q)(x-r)$. Then

$P(1) = (1-p)(1-q)(1-r) = 12$. Therefore, $\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right) = \frac{(p-1)(q-1)(r-1)}{pqr}$

$$= \frac{-P(1)}{4} = -\frac{12}{4} = -3.$$

23. A Construct $\overline{AD}$ and $\overline{DE}$. Using the law of cosines on ΔABC,



$$37^2 = 7^2 + 33^2 - 2(7)(33)\cos\angle ABC$$

From which, $\cos\angle ABC = -\frac{1}{2}$. Thus,

$m\angle ABC = 120°$. Therefore, ΔABD is equilateral, so that AD = BD = 7 and DC = 40.

Using the law of cosines on ΔADC, $40^2 = 7^2 + 37^2 - 2(7)(37)\cos\angle DAC$,

from which $\cos\angle DAC = -\frac{13}{37}$. Now, Using the law of cosines on ΔDAE,

$DE^2 = 7^2 + 7^2 - 2(7)(7)\left(-\frac{13}{37}\right) = 98 + \frac{(98)(13)}{37} = \frac{4900}{37}$, from which $DE = \frac{70}{\sqrt{37}}$.

24. B $1 + 2 + 3 + \ldots + 48 = \frac{(48)(49)}{2} = (24)(49)$. Let n = the number of integers from A to B, inclusive. Thus $n < 48$ and the sum of the integers from A to B, inclusive, is $\frac{n}{2}(A + B)$.

Therefore, $6\left[\frac{n}{2}(A + B)\right] = (24)(49) \Rightarrow 3n(A + B) = (2^3)(3)(7^2) \Rightarrow n(A + B) = (2^3)(7^2)$

Since $n < 48$, the only possible values of n are 2, 4, 7, 8, 14, and 28.

Letting n equal each of these values leads to the following results

$n = 2$	$A + B = 196$	which is impossible since $A, B < 48$
$n = 4$	$A + B = 98$	which is also impossible.
$n = 7$	$A + B = 56$	in this case, $B = A + 6$. These two equations give $A = 25$ and $B = 31$.
$n = 8$	$A + B = 49$	in this case, $B = A + 7$. These two equations give $A = 21$ and $B = 28$.
$n = 14$	$A + B = 28$	in this case, $B = A + 13$. These two equations give $A = 7\frac{1}{2}$ (not an integer).
$n = 28$	$A + B = 14$	in this case, $B = A + 27$. These two equations give a negative value for A.

Therefore, the only two solutions are (25, 31) and (21, 28).

25. B The numbers m and n cannot have a common factor $p > 1$, since in that case, any quantity not divisible by p is impossible, and there are infinitely many of these, not 35, as required. If we look at all the amounts represented by s schillings and f forinths, where $0 \leq s < m$ and $0 \leq f < n$, then no two of these mn amounts can be equal, nor can two differ by a multiple of mn. Thus, of all the amounts possible, every number of newtons is possible if s and f are permitted to be any number, positive or negative. Increasing s by m and decreasing f by n leaves a number unchanged. Similarly, decreasing s by m and increasing f by n leaves a number unchanged. Therefore, each quantity above mn newtons can be written with s and f positive. Of the mn values mentioned above, half of those with $0 < f < n$ lie below mn since $(m - s)n + (n - f)m = 2(m)(n) - (sn + fm)$.
The number of values with $s = 0$ or $f = 0$ is $n + m - 1$. Thus, the number of values between 0 and mn which are among the mn values is $(n + m - 1) + ½(m - 1)(n - 1)$.
Subtracting from mn, we have

$$mn - [(n + m - 1) + ½(m - 1)(n - 1)] = 35 \quad \text{or} \quad (m - 1)(n - 1) = 70.$$

The four possibilities for m and n are 2 and 71, 3 and 36, 6 and 15, and 8 and 11. The first allows 58, and the next two pairs have a common factor of 3.
Therefore, $m = 8$, $n = 11$ and the answer is (8,11).