THE 2019-2020 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

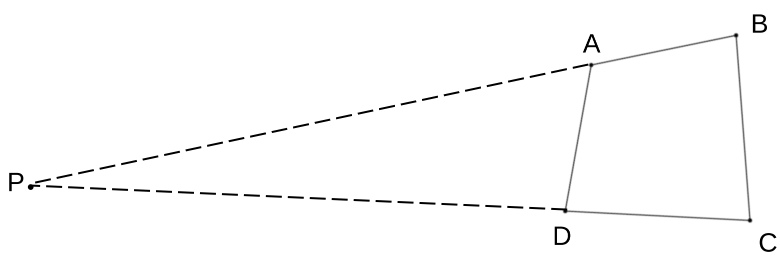
Calculators are NOT permitted

Time allowed: 2 hours

1. Four numbers are in arithmetic sequence. If the second number is decreased by 3 and the fourth number is increased by 12, the four numbers, in the same order, would then be in geometric sequence. Find all possible four number arithmetic sequences and prove that you have found them all.

2. Let $f(x) = x^2 + ax + b$, where a and b are integers. If $|f(0)| \leq 45^2$ and $f(300)$ is a prime number, prove that $f(x) = 0$ has no integer solutions.

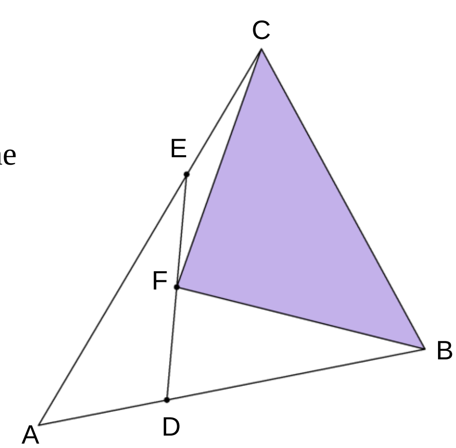
3. In quadrilateral ABCD, AB = AD = 4, BC = CD = 5, and $\angle ADC \cong \angle BCD$. $\overline{BA}$ and $\overline{CD}$ are extended to meet at point P. Compute, with proof, the distance from P to $\overline{BC}$.



4. The product $n(n + 13)$ is a perfect square when $n = 36$, since $36(36 + 13) = 1764 = 42^2$. In fact, $n = 36$ is the only value of n for which $n(n + 13)$ is a perfect square.

   Prove that for each prime number $p > 2$, there is exactly one positive integer n such that $n(n + p)$ is a perfect square.

5. In the diagram, AE = 2(EC), BD = 2(AD), and point F is the midpoint of $\overline{DE}$. Compute, with proof, the ratio of the area of triangle BFC to the area of triangle ABC.



Solutions

1. Method 1: Represent the four numbers in arithmetic sequence as a, $a + d$, $a + 2d$, and $a + 3d$. Then, the geometric sequence is a, $a + d – 3$, $a + 2d$, and $a + 3d + 12$. Therefore,

$$\frac{a+d-3}{a} = \frac{a+2d}{a+d-3} \Rightarrow (a+d-3)^2 = a(a+2d) \Rightarrow d^2 = 6a + 6d - 9 \quad (1)$$

Similarly, $\frac{a+3d+12}{a+2d} = \frac{a+2d}{a+d-3} \Rightarrow d^2 = 9a + 3d - 36$.

Therefore, $6a + 6d - 9 = 9a + 3d - 36 \Rightarrow a = d + 9$.
Substituting this last equation into (1) and simplifying,

$$d^2 - 12d - 45 = 0 \Rightarrow (d-15)(d+3) = 0 \Rightarrow d = 15, d = -3.$$

If $d = 15$, $a = 24$, and the arithmetic sequence is 24, 39, 54, 69
If $d = –3$, $a = 6$, and the arithmetic sequence is 6, 3, 0, –3.

A quick check shows that 24, 39, 54, 69 satisfies the conditions of the problem, with the corresponding geometric sequence being 24, 36, 54, 81.
However, 6, 3, 0, –3 does not work since 6, 0, 0, 9 is not a geometric sequence.
Therefore, the only arithmetic sequence is 24, 39, 54, 69.

Method 2: Represent the four numbers in arithmetic sequence as a, $a + d$, $a + 2d$, and $a + 3d$. Let the terms of the geometric sequence be represented by a, ar, ar^2, ar^3. Then

(1) $ar = a + d – 3$ and (2) $ar^2 = a + 2d$ and (3) $a + 3d = ar^3 - 12$

From (1) $a(r – 1) = d – 3$. From (2) $a(r^2 - 1) = 2d \Rightarrow a(r – 1)(r + 1) = 2d$.

Therefore, $(d – 3)(r + 1) = 2d \Rightarrow$ (4) $r + 1 = \frac{2d}{d-3}$.

From (3) $ar^3 - a = 3d + 12 \Rightarrow a(r^3 - 1) = 3d + 12 \Rightarrow a(r – 1)(r^2 + r + 1) = 3(d + 4)$

Substituting (1) into this last equation and dividing by $d – 3$, we obtain
(5) $(r^2 + r + 1) = \frac{3(d+4)}{d-3}$.

Substituting (4) into (5) we obtain $r^2 + \frac{2d}{d-3} = \frac{3(d+4)}{d-3} \Rightarrow r^2 = \frac{3(d+4)}{d-3} - \frac{2d}{d-3} = \frac{d+12}{d-3}$.

From (4) $r = \frac{2d}{d-3} - 1 = \frac{d+3}{d-3} \Rightarrow r^2 = \frac{(d+3)^2}{(d-3)^2}$.

Therefore, $\frac{d+12}{d-3} = \frac{(d+3)^2}{(d-3)^2}$ from which we eventually obtain $d = 15$. Thus, from (4), $r = \frac{3}{2}$ and from (1) $a = 24$. Therefore, the only such arithmetic sequence is 24, 39, 54, 69.

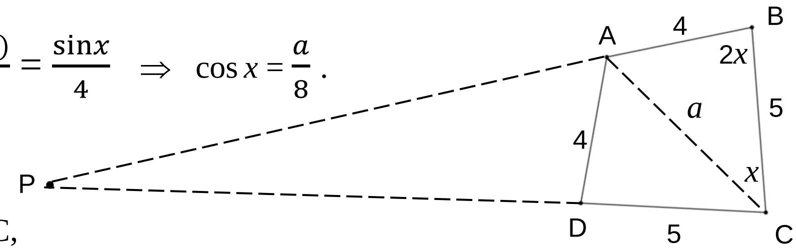
2. Assume that $f(x) = 0$ has an integer root x_1. Since the lead coefficient of $x^2 + ax + b$ is 1, the sum of the roots is $-a$. Since a is an integer, $f(x)$ has another integer root $x_2 = -a - x_1$. Thus, $f(x) = (x - x_1)(x - x_2)$, and $f(300) = (300 - x_1)(300 - x_2)$. Without loss of generality, let $x_1 > x_2$.

Since we are given $f(300)$ is prime, this means that $(300 - x_1) = \pm 1$ and $(300 - x_2)$ is prime. Therefore, $x_1 \geq 299$ while $|x_2| \geq 7$ (since 293 and 307 are the closest primes to 300). Since the product of the roots of $f(x) = x^2 + ax + b$ is b, $|f(0)| = |b| = |x_1 x_2| \geq 7 \cdot 299 = 2093$. But this is a contradiction, since we are given $|f(0)| \leq 45^2 < 2093$.

Therefore, $f(x) = 0$ has no integer solutions.

3. Method 1: Construct diagonal $\overline{AC}$. Since $\Delta ADC \cong \Delta ABC$ (SSS), $\angle ADC \cong \angle ABC$. Therefore, $m\angle DCA = m\angle BCA = \frac{1}{2} m\angle ABC$. Let $m\angle BCA = x$ and $m\angle ABC = 2x$, and let $AC = a$. Using the Law of Sines on ΔABC,

$$\frac{\sin 2x}{a} = \frac{\sin x}{4} \Rightarrow \frac{2(\sin x)(\cos x)}{a} = \frac{\sin x}{4} \Rightarrow \cos x = \frac{a}{8}.$$



Using the Law of Cosines on ΔABC,

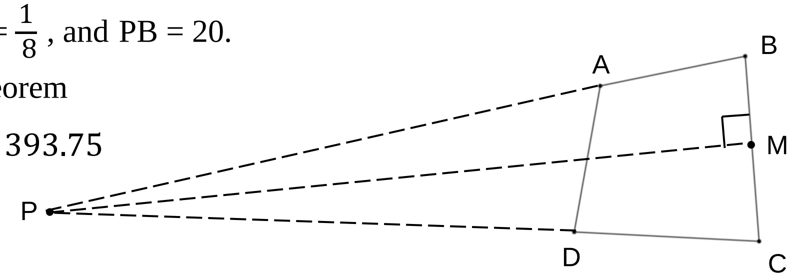
$$4^2 = 5^2 + a^2 - 2(5)(a)\left(\frac{a}{8}\right) \Rightarrow a = 6. \text{ Then } \cos x = \frac{a}{8} = \frac{3}{4}, \text{ and } \cos 2x = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}.$$

Now, construct the altitude of ΔPBC to $\overline{BC}$, meeting $\overline{BC}$ at point M. Since ΔPBC is isosceles ($\angle C \cong \angle B$), M is the midpoint of $\overline{BC}$. Thus, $BM = MC = 2.5$.

Then, in right ΔPMB, $\cos B = \frac{2.5}{PB} = \frac{1}{8}$, and $PB = 20$.

Finally, using the Pythagorean Theorem on ΔPMB, $PM^2 = 20^2 - 2.5^2 = 393.75$

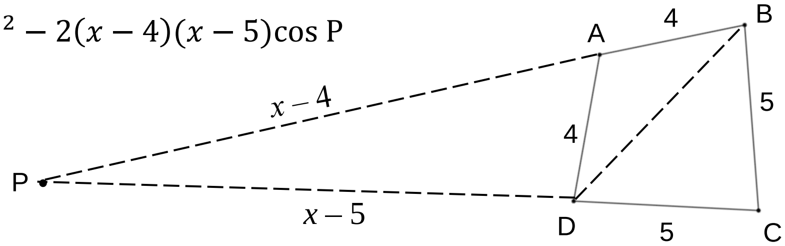
and $PM = \sqrt{393.75}$, or $\frac{15\sqrt{7}}{2}$, which is the desired distance.



Method 2: Construct diagonal $\overline{BD}$. Since ΔADB and ΔCDB are both isosceles triangles, $\angle ADC \cong \angle ABC$ and both are congruent to $\angle BCD$. Thus, ΔPBC is isosceles. Let $PB = x$, $PA = x - 4$, and $PD = x - 5$.

Using the Law of Cosines on ΔPAD,

(1) $16 = (x-4)^2 + (x-5)^2 - 2(x-4)(x-5)\cos P$



Using the Law of Cosines on ΔPBC,

$25 = x^2 + x^2 - 2x^2\cos P \Rightarrow \cos P = \frac{2x^2-25}{2x^2}$.

Substituting into (1) above,

$16 = 2x^2 - 18x + 41 - 2(x^2 - 9x + 20)\left(\frac{2x^2-25}{2x^2}\right)$.

Carefully simplifying this last equation, we obtain $2x^2 - 45x + 100 = 0$

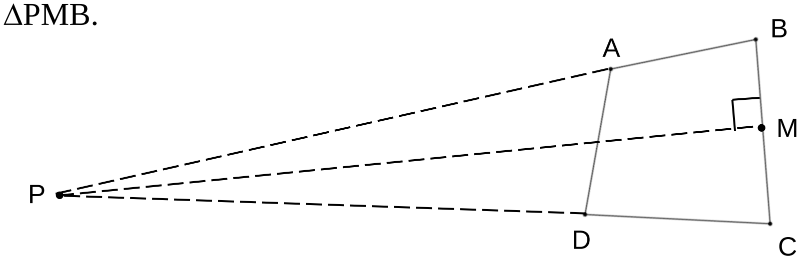
Factoring, $(2x-5)(x-20) = 0$ from which $x = \frac{5}{2}$ (impossible) and $x = 20$.

Finally, construct the altitude of PM of ΔPBC and noting that M is the midpoint of BC, use the Pythagorean Theorem on ΔPMB.

$\text{PM}^2 = 20^2 - 2.5^2 = 393.75$

and PM = $\sqrt{393.75}$, or $\frac{15\sqrt{7}}{2}$,

which is the desired distance.



4. Assume that $n(n+p) = a^2$ for some positive integer a.

We first prove that n is not a multiple of p.

Suppose that $n = kp$ for some integer k. Then $n + p = kp + p = (k+1)p$ and, therefore,

$$a^2 = n(n+p) = p^2k(k+1)$$

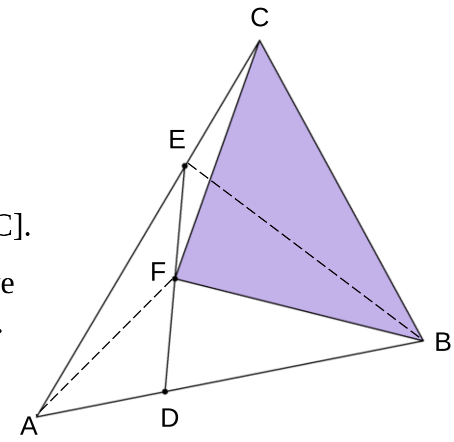
Hence, p must divide a which means $\frac{a}{p}$ is an integer, and $k(k+1) = \left(\frac{a}{p}\right)^2$.

Then, $k < \frac{a}{p} < k + 1$, which is impossible. Therefore, n is not a multiple of p.

Next, we prove that n and $n + p$ have no common prime factors. Suppose a prime q divides both n and $n + p$. Then q divides $(n + p) - n = p$, and $p = q$. But we already know that p does not divide n. So n and $n + p$ have no common prime factors.

Since $a^2 = n(n+p)$, and n and $n + p$ have no common prime factors, both n and $n + p$ must be perfect squares. Let $n + p = u^2$ and $n = v^2$ for some integers u and v. Then $p = u^2 - v^2 = (u+v)(u-v)$. Since p is prime, $u + v = p$ and $u - v = 1$. Subtracting these two equations, and solving for v, we get $v = \frac{p-1}{2}$ and $n = v^2 = \left(\frac{p-1}{2}\right)^2$, which is an integer since $p - 1$ is even. This is the only possible value of n for which $n(n+p)$ could be a square. Also, $n + p = \left(\frac{p-1}{2}\right)^2 + p = \frac{p^2+2p+1}{4} = \left(\frac{p+1}{2}\right)^2$ is the square of an integer, and so the product $n(n+p)$ is a square.

5. The desired ratio is $\frac{1}{2}$.



Method 1

Construct $\overline{AF}$ and $\overline{BE}$. Represent the area of ΔABC as $[\Delta ABC]$.

$[\Delta AFE] = [\Delta AFD]$, since $DF = FE$, and ΔAFE and ΔAFD have the same altitude from point A. Similarly, $[\Delta BFE] = [\Delta BFD]$.

Thus, $[\Delta AEB] = 2[\Delta AFB]$,

$[\Delta AFE] = 2[\Delta EFC]$, since $AE = 2(EC)$ and ΔAFE and ΔEFC have the same altitude from point F. Similarly, $[\Delta BFD] = 2[\Delta AFD] = 2[\Delta AFE] = 4[\Delta EFC]$.

Also, $[\Delta ADE] = [\Delta AFD] + [\Delta AFE] = 4[\Delta EFC]$.

$[\Delta AEB] = \frac{2}{3}[\Delta ABC]$, since $AE = \frac{2}{3}AC$ and the triangles have the same altitude from point B.

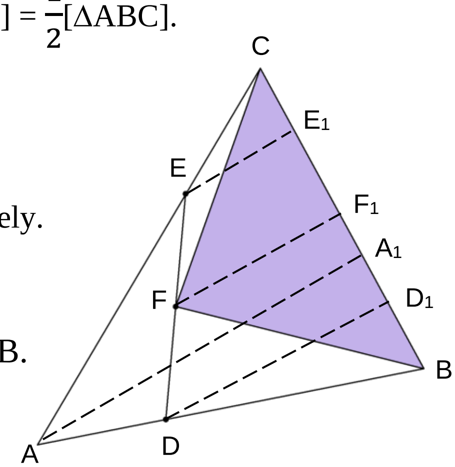
Therefore, $[\Delta AEB] = 2[\Delta AFB] = \frac{2}{3}[\Delta ABC] \Rightarrow [\Delta AFB] = \frac{1}{3}[\Delta ABC]$

Also, $[\Delta AFB] = [\Delta AFD] + [\Delta BFD] = [\Delta AFE] + [\Delta BFD]$
$= 2[\Delta EFC] + 2[\Delta AFD] = 2[\Delta EFC] + 4[\Delta EFC] = 6[\Delta EFC]$.

Therefore, $[\Delta AFB] = \frac{1}{3}[\Delta ABC] = 6[\Delta EFC] \Rightarrow [\Delta EFC] = \frac{1}{18}[\Delta ABC]$.

Finally, $[\Delta BFC] = [\Delta ABC] - [\Delta EFC] - [\Delta ADE] - [\Delta BFD]$

$$= [\Delta ABC] - \frac{1}{18}[\Delta ABC] - \frac{4}{18}[\Delta ABC] - \frac{4}{18}[\Delta ABC] = \frac{1}{2}[\Delta ABC].$$



Method 2

Construct perpendiculars from D, A, F, and E to $\overline{BC}$, and label the points of intersection D_1, A_1, F_1, and E_1, respectively.

The area of $\Delta ABC = \frac{1}{2}(BC)(AA_1)$

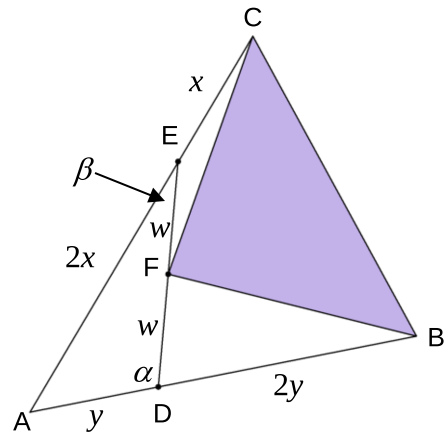
Since DD_1 is parallel to AA_1, ΔDD_1B is similar to ΔAA_1B.
Therefore, $\frac{DD_1}{AA_1} = \frac{DB}{AB} = \frac{2}{3} \Rightarrow DD_1 = \frac{2}{3}AA_1$.

Similarly, ΔEE_1C is similar to ΔAA_1C, and $\frac{EE_1}{AA_1} = \frac{EC}{AC} = \frac{1}{3} \Rightarrow EE_1 = \frac{1}{3}AA_1$.

Since F is the midpoint of $\overline{ED}$, FF_1 is the median of trapezoid EE_1D_1D.
Then, $FF_1 = \frac{1}{2}(EE_1 + DD_1) = \frac{1}{2}(\frac{1}{3}AA_1 + \frac{2}{3}AA_1) = \frac{1}{2}AA_1$.

Thus, the area of $\Delta BCF = \frac{1}{2}(BC)(FF_1) = \frac{1}{2}(BC)(\frac{1}{2}AA_1) = \frac{1}{2}\left[\frac{1}{2}(BC)(AA_1)\right]$.
Therefore, the area of ΔBCF is half the area of ΔABC.

Method 3



Let EC = x, EA = $2x$, AD = y, BD = $2y$, and DF = EF = w.

Let $\angle$ADE = α and $\angle$AED = β.

Area ΔABC = $\frac{1}{2}(3x)(3y)\sin A = \frac{9xy\sin A}{2}$.

Area ΔAED = $\frac{1}{2}(2x)(y)\sin A = xy\sin A$.

Area ΔEFC = $\frac{1}{2}xw\sin(180-\beta) = \frac{1}{2}xw\sin\beta$.

Area ΔFDB = $\frac{1}{2}w(2y)\sin(180-\alpha) = yw\sin\alpha$.

Using the Law of Sines on ΔAED, $\frac{2x}{\sin\alpha} = \frac{2w}{\sin A} = \frac{y}{\sin\beta}$.

Therefore, $\sin\beta = \frac{y\sin A}{2w}$ and $\sin\alpha = \frac{x\sin A}{w}$.

Then Area ΔEFC = $\frac{1}{2}xw\left(\frac{y\sin A}{2w}\right) = \frac{xy\sin A}{4}$, and Area ΔFDB = $yw\left(\frac{x\sin A}{w}\right) = xy\sin A$.

Therefore,

$$\frac{\text{Area}\Delta\text{BFC}}{\text{Area}\Delta\text{ABC}} = \frac{\text{Area}\Delta\text{ABC} - \text{Area}\Delta\text{AED} - \text{Area}\Delta\text{EFC} - \text{Area}\Delta\text{FDB}}{\text{Area}\Delta\text{ABC}} =$$

$$\frac{\frac{9xy\sin A}{2} - xy\sin A - \frac{xy\sin A}{4} - xy\sin A}{\frac{9xy\sin A}{2}} = \frac{\left(\frac{9}{2} - 1 - \frac{1}{4} - 1\right)xy\sin A}{\frac{9}{2}xy\sin A} = \frac{1}{2}.$$