THE 2021-2022 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

Calculators are NOT permitted

Time allowed: 2 hours

1. Each of three cards has an integer written on it. The three integers p, q, r satisfy the condition $0 \le p < q < r$. Three players A, B, C mix the cards and pick one each. They record the number on their card, mix the cards and pick one each again. The number on the card they select is added to their previous number. This process is repeated at most ten times, after which A has 20 points, B has 10 points, and C has 9 points. If B got the r card in the last round, determine, with proof, who received the q card in the first round.

2. Most cubic polynomials have either three distinct real roots or just one real root. Sometimes, however, a cubic has exactly two distinct real roots because two of the three roots coincide. Find, with proof, all real numbers a such that the polynomial $x^3 - 7x^2 + ax - 9$ has exactly two distinct real roots.

3. Quadrilateral ABCD has the following properties:

   (i) The center of the semicircle, O, is the midpoint of side $\overline{AB}$.

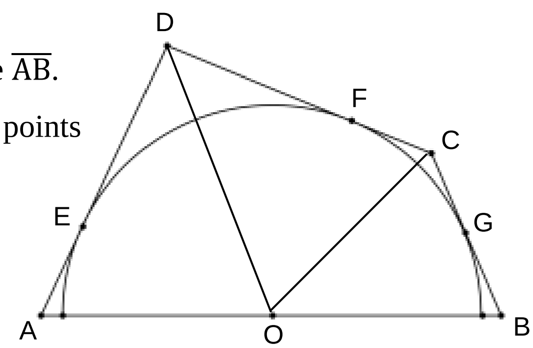
   (ii) Sides $\overline{AD}$, $\overline{DC}$, and $\overline{CB}$ are tangent to the semicircle at points E, F, and G, respectively.

   Line segments $\overline{OD}$ and $\overline{OC}$ divide the quadrilateral into three triangles.

   (a) Prove that these three triangles are similar.

   (b) Prove that $(AB)^2 = 4(AD)(BC)$.

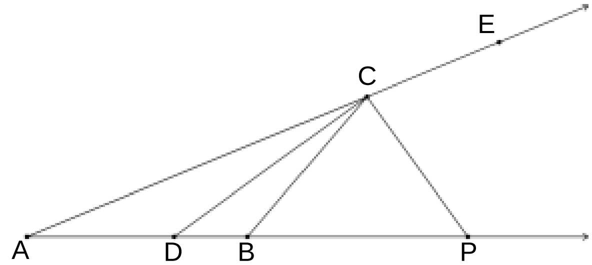
   (**Please include a diagram with your proofs.**)



4. The number 210 has an interesting property. It can be written both as the product of two consecutive integers $(14 \cdot 15)$ and also the product of three consecutive integers $(5 \cdot 6 \cdot 7)$. Prove that there exists no positive integer that can be written both as the product of two consecutive integers and also four consecutive integers.

5. In triangle ABC, AB = 6, AC =10 and BC = 5. Side $\overline{AC}$ is extended through point C and the angle bisectors of angle ACB and BCE are constructed, intersecting ray AB in points D and P, respectively, as shown. Compute, with proof, the length $\overline{PD}$.

   (**Please include a diagram with your proof.**)



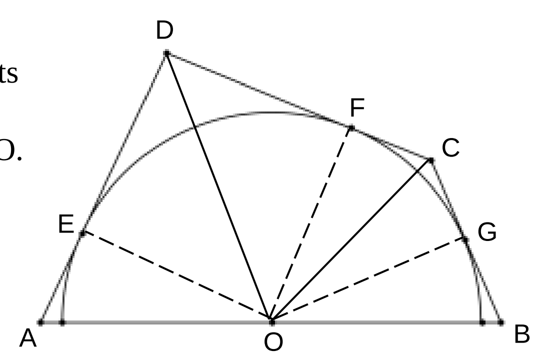
Solutions

1. Because a constant number of points, $p + q + r$, is awarded in each round, $p + q + r$ must divide the total number of points, 39. It follows that the number of rounds must be 1, 3, 13 or 39. As there were at most 10 rounds, there must have been 1 or 3. There can't have been only one round because B would have gotten the r card in that round making B's score greater than A's score. Since this is not possible, there were exactly three rounds, and $p + q + r = 13$. Observe that no player got all three of p, q and r since their total is 13. Thus, each player has a duplication. Note also that $r \leq 10$ because B got a total of 10, and $r \geq 7$, since the total for A cannot reach 20 if $r < 7$. We have the following possibilities to check:

p	0	1	0	1	0	1	2	0	1	2
q	3	2	4	3	5	4	3	6	5	4
r	10	10	9	9	8	8	8	7	7	7

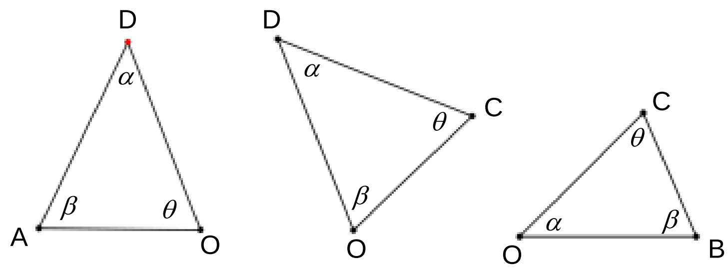
The only triples (p, q, r) that allow A to get 20 points are (0, 3, 10) and (1, 4, 8). In the first case, A must have drawn 10, 10, 0, B must have drawn 0, 0, 10, and C 3, 3, 3. In the second case, A must have drawn 8, 8, 4, B must have drawn 1, 1, 8, and C 4, 4, 1. In each of our two solutions, C drew the q card on the first round.

2. Suppose the polynomial $x^3 - 7x^2 + ax - 9$ has a repeated root r and some other root s. It is well known that the coefficient -7 of x^2 is the negative of the sum of the three roots and that the constant term -9 is the negative of the product of the three roots (Vieta's Theorem). Thus, $2r + s = 7$ and $r^2 s = 9$, so $s = 7 - 2r$ and $9 = r^2 s = r^2(7 - 2r)$. This yields the equation $2r^3 - 7r^2 + 9 = 0$, and while it is often difficult to solve a cubic equation, it is easy to see that $r = -1$ is a solution of this one. Thus, we can factor $2r^3 - 7r^2 + 9 = (r + 1)P(r)$ for some polynomial $P(r)$. Then dividing $2r^3 - 7r^2 + 9$ by $r + 1$ we obtain $P(r) = 2r^2 - 9r + 9$, so to find the other possible values for r, we must solve the quadratic equation $2r^2 - 9r + 9 = 0$. Factoring, we find the two solutions $r = 3$ and $r = \frac{3}{2}$, and thus the three possibilities for r are -1, 3 and $\frac{3}{2}$, and the corresponding values for $s = 7 - 2r$ are 9, 1 and 4. In each of these cases, the original cubic has exactly two distinct roots. To find the corresponding values of a, substitute each value of r into $x^3 - 7x^2 + ax - 9 = 0$ and solve for a. If $r = -1$, then $a = -17$; if $r = 3$, then $a = 15$; and if $r = \frac{3}{2}$, then $a = \frac{57}{4}$. Thus, there are three values of a, namely -17, 15, and $\frac{57}{4}$.



3. (a) Construct radii OE, OF, and OG. Since the tangent segments to a circle from an external point are congruent, $\overline{DE} \cong \overline{DF}$ and $\overline{CF} \cong \overline{CG}$. Since ΔDEO $\cong$ ΔDFO (SSS), $\angle$EDO $\cong$ $\angle$FDO. Similarly, ΔCFO $\cong$ ΔCGO, and $\angle$FCO $\cong$ $\angle$GCO.
Let m$\angle$EDO = m$\angle$FDO = α, and m$\angle$FCO = m$\angle$GCO = θ.
Also, ΔAEO $\cong$ ΔBGO (HL), so that $\angle$A $\cong$ $\angle$B.
Let m$\angle$A = m$\angle$B = β.

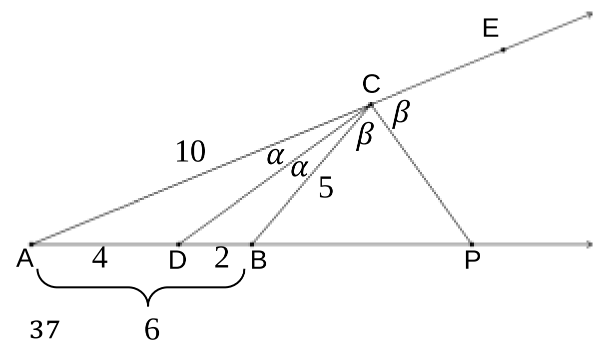
Adding the measures of the angles of quadrilateral ABCD, $\beta + 2\alpha + 2\theta + \beta = 360°$. Hence $\alpha + \theta + \beta = 180°$ and thus they are measures of the angles of a triangle. Since two of the angle measures in ΔAOD are β and α, the third angle measure must be θ. Similarly, the angles of ΔOCD and ΔBCO have the same three measures (shown below), making all three triangles similar (ΔAOD ~ ΔOCD ~ ΔBCO).



(b) Since ΔAOD ~ ΔBCO, $\frac{\text{AD}}{\text{OB}} = \frac{\text{AO}}{\text{BC}}$, or (AD)(BC) = (AO)(OB). Since AO = OB = ½ (AB), we get $(\text{AB})^2 = 4(\text{AD})(\text{BC})$.

4. Suppose we could write a number both as $n(n + 1)$ and as $m(m + 1)(m + 2)(m + 3)$, where n and m are positive integers. Multiplying the outer and inner factors of the second expression gives $(m^2 + 3m)(m^2 + 3m + 2)$. Letting $k = m^2 + 3m$, we have $n(n + 1) = k(k + 2)$. Adding 1 to each side, $n^2 + n + 1 = k^2 + 2k + 1 = (k + 1)^2$. So $n^2 + n + 1$ is a perfect square. But $n^2 < n^2 + n + 1 < (n + 1)^2 = n^2 + 2n + 1$. Thus, $n^2 + n + 1$ is a perfect square that lies between 2 consecutive perfect squares, which is impossible. Therefore, there are no integers that can be written both as the product of two and also four consecutive positive integers.

5. Method 1



In ΔABC, since $\overline{\text{CD}}$ bisects ∠ACB, $\frac{10}{\text{AD}} = \frac{5}{6-\text{AD}}$, from which AD = 4 and DB = 2.

Using the Law of Cosines on ΔABC,

$$5^2 = 10^2 + 6^2 - 2(10)(6)\cos \text{A} \Rightarrow \cos \text{A} = \frac{37}{40}.$$

Using the Law of Cosines on ΔADC,

$$\text{CD}^2 = 10^2 + 4^2 - 2(10)(4)\frac{37}{40} \Rightarrow \text{CD} = \sqrt{42}.$$

Using the Law of Cosines on ΔADC again,

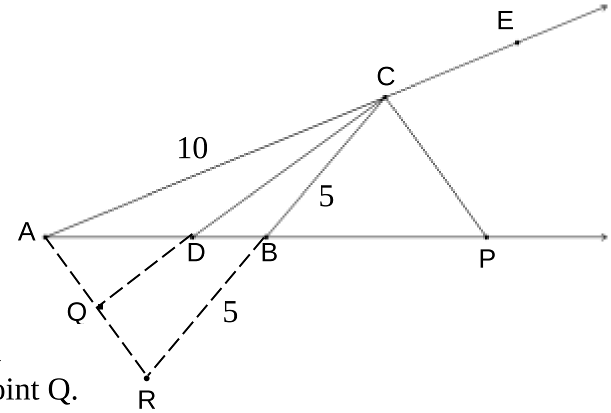
$$10^2 = 4^2 + \left(\sqrt{42}\right)^2 - 2(4)\left(\sqrt{42}\right)\cos \angle\text{ADC} \Rightarrow \cos \angle\text{ADC} = \frac{-\sqrt{42}}{8}.$$

Since ∠CDP is supplementary to ∠ADC, $\cos \angle\text{CDP} = \frac{\sqrt{42}}{8}$.

Let m∠ACD = m∠DCB = α, and m∠BCP = m∠PCE = β.

Then, $2\alpha + 2\beta = 180$ and $\alpha + \beta = 90$, making ∠DCP a right angle.

Using right triangle CDP, $\cos \angle\text{CDP} = \frac{\sqrt{42}}{\text{DP}}$. Therefore, DP = 8.



Method 2

From Method 1, AD = 4 and DB = 2, and $\angle$DCP is a right angle.

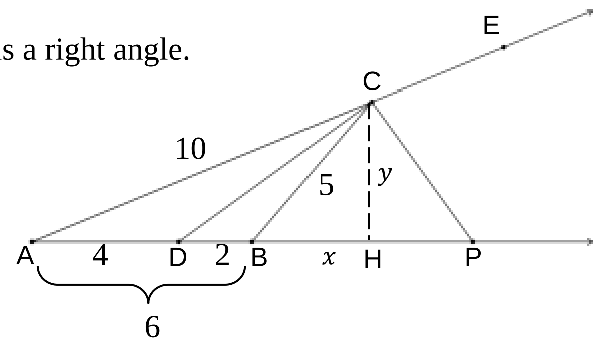
Extend $\overline{CB}$ its own length through B to a point R, and construct $\overline{AR}$. Extend $\overline{CD}$ through D to meet $\overline{AR}$ at point Q.

Since CR = 10, ΔACR is isosceles, and since $\overline{CQ}$ bisects vertex angle ACR, $\overline{CQ}$ is also an altitude in ΔACR. Then $\overline{AR}$ is parallel to $\overline{CP}$ since $\overline{CQ}$ is perpendicular to both. Therefore, $\angle$BCP and $\angle$BRA are congruent alternate interior angles.

Then, ΔABR $\cong$ ΔPBC, so that BP = AB = 6, and DP = DB + BP = 8.

Method 3

From Method 1, AD = 4 and DB = 2, and $\angle$DCP is a right angle.



Construct the altitude of ΔABC from C, meeting Line AB at point H. Let BH = x and CH = y.

Using the Pythagorean Theorem on ΔACH,
$(x+6)^2 + y^2 = 10^2 \Rightarrow\ y^2 = 64 - 12x - x^2$

Using the Pythagorean Theorem on ΔDCH,
$(x+2)^2 + y^2 = 5^2 \Rightarrow\ y^2 = 21 - 4x - x^2$.

Setting the two equations equal and solving for x, we get $x = \frac{13}{4}$, from which $y = \frac{\sqrt{231}}{4}$.
Using the Pythagorean Theorem on ΔCDH, CD = $\sqrt{42}$.

Because right ΔDCP and right ΔDHC share $\angle$D, they are similar. Then
$\frac{\mathrm{DH}}{\mathrm{DC}} = \frac{\mathrm{DC}}{\mathrm{DP}} \Rightarrow\ \mathrm{DP} = \frac{(\mathrm{DC})^2}{\mathrm{DH}} = \frac{42}{\frac{21}{4}} = 8.$