KENNESAW STATE UNIVERSITY
COLLEGE OF SCIENCE AND MATHEMATICS
Department of Mathematics

THE 2023–2024 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

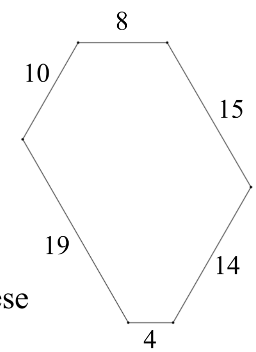
PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

1. A number is *Beprisque* if it is the only integer between a prime number and a perfect square. For example, 8 is *Beprisque* because it is the only integer between the prime 7 and the perfect square 9. Compute the number of *Beprisque* numbers less than 100.

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9



2. The diagram shows an equiangular hexagon with side lengths 4, 8, 10, 14, 15, and 19. The hexagon can be inscribed in an equilateral triangle so that three sides of the hexagon lie on the sides of the equilateral triangle. Compute the sum of the perimeters of all such equilateral triangles.

(A) 111 (B) 153 (C) 210 (D) 280 (E) None of these

3. If either the first or the last digit of a certain three-digit number is deleted, the remaining two-digit number is 8 times the deleted digit. If the middle digit is deleted, compute the ratio of the remaining two-digit number to the deleted middle digit.

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

4. The three-digit number $\overline{xyz}_7$ is equal to the three-digit number $\overline{zyx}_5$. Compute the sum of all possible values for the base 10 representation of the number. (Note: Numbers like $\overline{007}$ and $\overline{039}$ are not considered three-digit numbers.)

(A) 153 (B) 154 (C) 155 (D) 156 (E) 157

5. Chuck has somewhere between 1000 and 2000 Pokemon cards that he wants to divide into two or more piles, each with the same number of cards. He tries 2, 3, 4, 5, 6, 7, and 8 piles but ends up with one card left over each time. What is the minimum number of piles that he needs?

(A) 36 (B) 39 (C) 41 (D) 43 (E) 48

6. Don and Debbie live in a skyscraper with ten apartments on each floor. The apartments on the first floor are numbered 1 through 10, those on the second floor are numbered 11 through 20, and so on. Don's apartment is on a floor whose number is the same as the number of Debbie's apartment, and the sum of the numbers of their apartments is 239. What is the number of Don's apartment?

(A) 217 (B) 212 (C) 196 (D) 189 (E) None of these

7. Let S be the set of two–digit primes. If a number is chosen at random from S, what is the probability that the product of its digits is a prime number?

(A) $\frac{1}{5}$ (B) $\frac{1}{7}$ (C) $\frac{4}{21}$ (D) $\frac{5}{21}$ (E) $\frac{2}{7}$

8. Dr. Garner gave the nine students in his AP Statistics class a 10-question quiz, with each question worth one point. The mean score in the class was 8, the median was 8, and the mode was 7. Determine the maximum number of perfect scores (all 10 questions correct) possible on this quiz.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

9. Let r and s be the solutions to the equation $x^2 + 3x + c = 0$. If $r^2 + s^2 = 33$, compute the value of c.

(A) –21 (B) –12 (C) 1 (D) 12 (E) 21

10. A bag of coins contains pennies, dimes and quarters. There are twice as many dimes as pennies and three times as many quarters as dimes. Which of the following could be the amount of money in the bag?

(A) \$306 (B) \$333 (C) \$342 (D) \$348 (E) \$360

11. If $f(x) = \frac{x+1}{x-1}$, compute the value of x for which $f^{-1}\left(\frac{1}{x}\right) = 21$.

(A) $\frac{11}{10}$ (B) $\frac{10}{11}$ (C) $\frac{6}{5}$ (D) $\frac{5}{6}$ (E) None of these

12. On hypotenuse $\overline{AB}$ of right triangle ABC, square ABDE is drawn externally. If AC = 8 and BC = 6 and the length of CD is $\sqrt{k}$, compute k.

(A) 208 (B) 232 (C) 240 (D) 252 (E) None of these

13. One of the roots of the polynomial equation $x^4 + ax^2 + b = 0$ is $\sqrt{2} + \sqrt{5}$. If a and b are rational numbers, compute the value of $b - a$.

(A) 3 (B) 7 C) 21 (D) 23 (E) 29

14. Compute the value of $\left(\log_{2023}\frac{1}{2}\right)\left(\log_{2022}\frac{1}{3}\right)\left(\log_{2021}\frac{1}{4}\right)\cdots\left(\log_{3}\frac{1}{2022}\right)\left(\log_{2}\frac{1}{2023}\right)$.

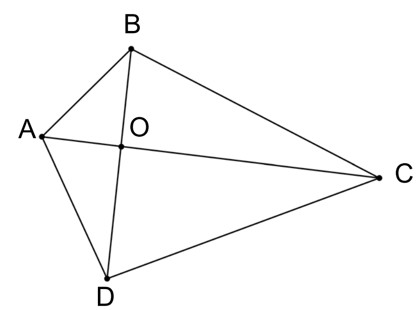
(A) –2023 (B) –1 (C) 1 (D) 2023 (E) None of these

15. If $c^2 = (a + bi)^2 - 130i$, where i is the imaginary unit and a, b, and c are positive integers, compute c.

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

16. The first two positive integers n for which $1 + 2 + 3 +\ldots+ n$ is a perfect square are 1 and 8. Compute the sum of the next two.

(A) 337 (B) 359 (C) 401 (D) 443 (E) 518



17. Let ABCD be the convex quadrilateral, as shown, and let O be the point of intersection of its two diagonals. Suppose the area of ΔABD is 1, the area of ΔBCA is 2 and the area of ΔDAC is 3. Compute the area of ΔABO.

(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{3}{8}$ (D) $\frac{5}{12}$ (E) None of these

18. Five positive real numbers, a_1, a_2, a_3, a_4, and a_5 are such that a_1, a_2, a_3 are in arithmetic sequence, a_2, a_3, a_4 are in geometric sequence, and a_3, a_4, a_5 are in harmonic sequence (i.e. their reciprocals are in arithmetic sequence). If $a_1 = 1$ and $a_5 = 5$, compute a_3.

(A) $\sqrt{5}$ (B) $1 + \sqrt{5}$ (C) $\frac{\sqrt{5}}{2}$ (D) $\frac{1+\sqrt{5}}{2}$ (E) None of these

19. In quadrilateral ABCD, the measures of ∠A, ∠B, ∠C, and ∠D, in that order, form an arithmetic sequence. The circle through points B, C, and D is tangent to $\overline{AD}$ and the measure of minor arc BC is 44 degrees. Compute the degree measure of ∠A.

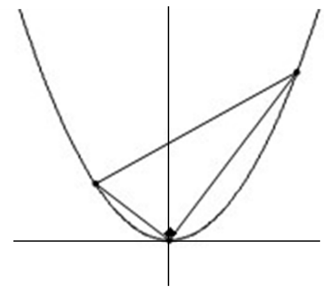
(A) 22 (B) 44 (C) 51 (D) 55 (E) 57

20. In the magic square shown, each row, column, and main diagonal sum to 100, where T, R, I, A, N, G, L, E are (not necessarily distinct) real numbers. Compute G + E + A + R.

T	R	I
A	N	G
L	E	2023

(A) $\frac{400}{3}$ (B) 400 (C) 2023 (D) $\frac{2023}{3}$ (E) 4046

21. Shown is a right triangle, with right angle at the origin, inscribed in the graph of the parabola $y = x^2$. If the area of the right triangle is $\frac{5}{3}$, compute the length of the longer leg of the triangle.



(A) $6\sqrt{2}$ (B) $5\sqrt{3}$ (C) $4\sqrt{6}$ (D) $3\sqrt{10}$ (E) $2\sqrt{15}$

22. A palindrome is a number that reads the same backwards and forwards. How many four-digit palindromes of the form $abba$, are divisible by the two-digit palindrome aa?

(A) 46 (B) 47 (C) 48 (D) 49 (E) 50

23. Let a be a real solution to the equation $x^3 + 4x = 8$. Compute the value of $a^7 + 64a^2$.

(A) 128 (B) 256 (C) 512 (D) 1024 (E) None of these

24. For all real $a \neq -1$, define $a^* = \frac{a-1}{a+1}$. Determine the value of n for which $(n^*)^* = \tan 15°$.

(A) $-2-\sqrt{3}$ (B) $1+\sqrt{3}$ (C) $1-2\sqrt{3}$ (D) $2+\sqrt{3}$ (E) $2-\sqrt{3}$

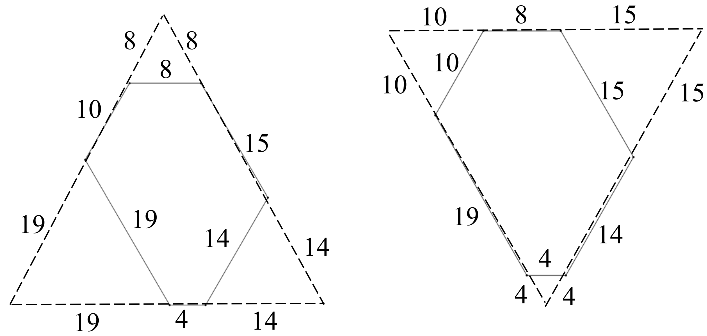
25. A "rising" number is a positive integer each digit of which is larger than each of the digits to its left, such as 34689. When all five-digit rising numbers are arranged from smallest to largest, what is the 100^{th} number in the list.

(A) 25789 (B) 25689 (C) 25679 (D) 25678 (E) 24789

Solutions

1. D Listing the perfect squares less than 100, it is easy to identify the *Beprisque* numbers as 2, 3, 8, 10, 24, 48, 80, 82, for a total of 8.

2. C There are 2 possibilities shown, with side lengths 37 and 33. The sum of the perimeters is 210.



3. D Let the three-digit number be $100a + 10b + c$. Then $10a + b = 8c$ and $10b + c = 8a$. Eliminating c and a from the two equations, respectively, we obtain $2a = 3b$ and $2b = c$. Therefore, $\dfrac{10a + c}{b} = \dfrac{15b + 2b}{b} = 17$. (The only possible three-digit numbers are 324 and 648.)

4. A $\overline{xyz_7} = 49x + 7y + z$ and $\overline{zyx_5} = 25z + 5y + x$. Setting the two equal and solving for y gives $y = 12z - 24x$. Noting that $0 \le x, y, z < 5$, the only possible values for $\overline{xyz_7}$ are 102 and 204. These numbers are 51 and 102 in base 10, and their sum is 153.

5. C The number of cards will leave a remainder of 1 when divided by the least common multiple of 2, 3, 4, 5, 6, 7, and 8, which is $8 \cdot 3 \cdot 5 \cdot 7 = 840$. Since the number of cards is between 1000 and 2000, the only possibility is 1681. The number of piles must be a divisor of $1681 = 41^2$, hence it must be 41.

6. A If a is the number of Don's floor, then the number of Don's apartment can be represented by $10(a - 1) + b$, where $1 \le b \le 10$. The given conditions imply that $10(a - 1) + b + a = 11a + b - 10 = 239$, or $11a = 249 - b$. Therefore $249 - b$ is divisible by 11. Because $1 \le b \le 10$, we can conclude that $249 - b = 242$. Thus,
$b = 7$ and $a = 22$. The number of Don's apartment is, therefore, $239 - 22 = 217$.

7. C $S = \{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$. Of these 21 numbers, only the product of the digits of 13, 17, 31, and 71 yield a prime. Therefore, the desired probability is $\frac{4}{21}$.

8. C There can't have been four or more 10's because the mode is 7, meaning there had to be more 7's than 10's (five 7's and four 10's doesn't average to 8). If there were three 10's there needs to be at least four 7's. There can't be five 7's because the median, the fifth largest number out of the 9, is an 8. If there are four 7's, the numbers would be $\{7,7,7,7,8,x,10,10,10\}$, with x either 8 or 9. However, neither of these sets of nine scores average to 8. Thus, there were at most two 10's, and a possible score distribution is $\{7,7,7,7,8,8,8,10,10\}$.

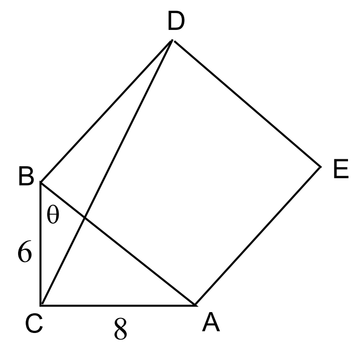
9. B Consider $(r + s)^2 = r^2 + 2rs + s^2$. We are given $r^2 + s^2 = 33$. Therefore,

$(r+s)^2 = 2rs + 33$. However, $rs = c$, and $r + s = -3$. Thus, $(-3)^2 = 2c + 33$ and $c = -12$

10. C Let p = number of pennies, $2p$ = number of dimes, and $6p$ = number of quarters. Then the value of the coins in cents is $p + 20p + 150p = 171p$. The amount of money in the bag must be divisible by 171. Only choice (C) works.

11. B Let $y = f^{-1}(x)$. Then, $x = \frac{y+1}{y-1}$ which is equivalent to $y = \frac{x+1}{x-1}$. So, it turns out that $f(x) = f^{-1}(x)$. This means $f\left(f^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{x} = f(21)$. Therefore, $x = \frac{21-1}{21+1} = \frac{10}{11}$.

12. B Using the Pythagorean Theorem, AB = BD = 10.



Note that $\sin\theta = \frac{8}{10}$ and $\cos(\angle CBD) = \cos(90+\theta) = -\sin\theta = -\frac{8}{10}$.

Use the Law of Cosines on triangle CBD.

$CD^2 = 6^2 + 10^2 - 2(6)(10)(-\frac{8}{10}) = 232$. Therefore, $k = 232$.

13. D <u>Method 1</u>: Let $x = \sqrt{2} + \sqrt{5}$. Then $x - \sqrt{2} = \sqrt{5}$ and $(x - \sqrt{2})^2 = 5$. From this, $x^2 - 2x\sqrt{2} + 2 = 5$ and $x^2 - 3 = 2x\sqrt{2}$. Squaring both sides, we obtain $x^4 - 6x^2 + 9 = 8x^2$ or $x^4 - 14x^2 + 9 = 0$. Therefore, $a = -14$, $b = 9$ and $b - a = 23$.

14. C Apply the change of base formula to each term and note that the power property implies $\log\frac{1}{x} = -\log x$. Thus, the expression becomes

$$\frac{-\log 2}{\log 2023} \cdot \frac{-\log 3}{\log 2022} \cdot \frac{-\log 4}{\log 2021} \cdots \frac{-\log 2022}{\log 3} \cdot \frac{-\log 2023}{\log 2} = (-1)^{2022} = 1$$

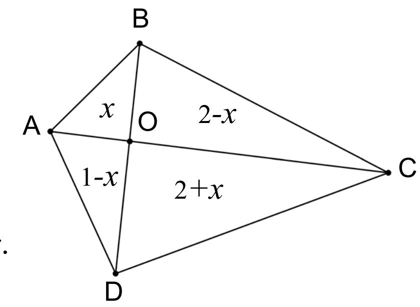
15. D Expanding the given equation and solving for c gives

$$c^2 = a^2 + 2abi - b^2 - 130i$$

Since c is a positive integer, $2abi - 130i = 0 \Rightarrow 2ab - 130 = 0 \Rightarrow ab = 65$.
Thus, either $a = 65$ and $b = 1$, or $a = 13$ and $b = 5$.
When $a = 65$ and $b = 1$, we have $c^2 = a^2 - b^2 = 65^2 - 1^2$ and c is not an integer.
When $a = 13$ and $b = 5$, we have $c^2 = 13^2 - 5^2 = 144$, and $c = 12$.

16. A Since $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, we need $n(n+1)$ to be twice a perfect square.
Since n and $n + 1$ have no common divisors, one must be a square and the other twice a square. The square must be odd. Examine the first few odd squares k^2 to see if $k^2 - 1$ or $k^2 + 1$ is 2 times a square. For $k^2 - 1$, we obtain 8, 24, 48, 80, 120, 168, 224, 288. For $k^2 + 1$, we obtain 10, 26, 50, 82, 122, 170, 226, 290. Of these, 8, 50, and 288 are 2 times a perfect square. Thus, 49 and 288 are the numbers we are looking for, with a sum of 337.

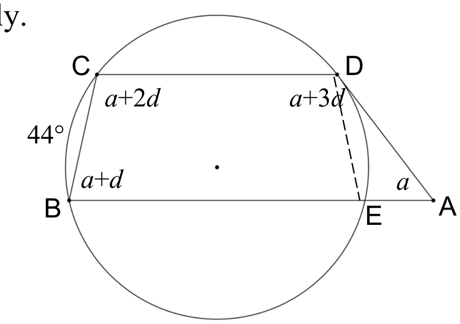
17. B Let x equal the area of ΔABO. Since the area of ΔBCA is 2, it follows that the area of ΔBCO is equal to $2 - x$. Similarly, since ΔABD has area 1, we see that ΔDAO has area equal to $1 - x$. Finally, since ΔDAC has area 3, we conclude that the area of ΔCDO is equal to $3 - (1 - x) = 2 + x$. In particular, the area of ΔCDB is $(2 - x) + (2 + x) = 4$. Now ΔABO and ΔBCO share the same altitude to AC, so their areas are proportional to the lengths of their bases, namely AO and OC. Similarly, the areas of ΔDAO and ΔCDO are also proportional to AO and OC. Thus, $\frac{\text{AO}}{\text{OC}} = \frac{x}{2-x} = \frac{1-x}{2+x} \Rightarrow 2x + x^2 = x^2 - 3x + 2$, from which x = area of ΔABO = $\frac{2}{5}$.



18. A Since a_1, a_2, a_3 are in arithmetic sequence, $a_3 - a_2 = a_2 - 1$ and $a_3 = 2a_2 - 1$. Therefore, $a_2, 2a_2 - 1, a_4$ are in geometric sequence which gives us or $a_4 = \frac{(2a_2-1)^2}{a_2}$. Now, $2a_2 - 1, \frac{(2a_2-1)^2}{a_2}, a_5$ are in harmonic sequence, which gives us $\frac{2a_2-1}{(2a_2-1)^2}, \frac{a_2}{(2a_2-1)^2}, \frac{1}{a_5}$ are in arithmetic sequence Therefore, $\frac{1}{a_5} = \frac{1}{(2a_2-1)^2}$ Hence $a_5 = 5 = (2a_2 - 1)^2 = (a_3)^2$. Thus, $a_3 = \sqrt{5}$.

19. E Let the circle intersect $\overline{\text{AB}}$ again at point E. Let the measures of angles A, B, C, and D be a, $a + d$, $a + 2d$, and $a + 3d$, respectively.

Then $4a + 6d = 360 \Rightarrow 2a + 3d = 180$. Thus, ∠B and ∠C are supplementary, making $\overline{\text{AB}}$ parallel to $\overline{\text{CD}}$. Construct chord $\overline{\text{DE}}$. Since parallel chords intercept congruent arcs, m($\overset{\frown}{\text{CB}}$) = m($\overset{\frown}{\text{DE}}$) = 44 and $\overline{\text{BC}} \cong \overline{\text{ED}}$.



Therefore, quadrilateral BCDE is an isosceles trapezoid with m∠BCD = m∠CDE = $a + 2d$. Thus, m∠ADE = d.

Since an angle formed by a tangent and a chord is half the measure of its intercepted arc, d = m∠ADE = 22. Hence, $2a + 3d = 2a + 66 = 180$, and a = m∠A = 57°.

20. A Sum the middle row and middle column to get

(i) 200 = (G+E+A+R) + 2N.

Another equation is needed to eliminate N, so sum all of the rows to get 300 = (G+E+A+R) + (T+I+N+L+2023) = (G+E+A+R) + (T+N+2023) + (I+N+L) – N = (G+E+A+R) + 200 – N. Thus,

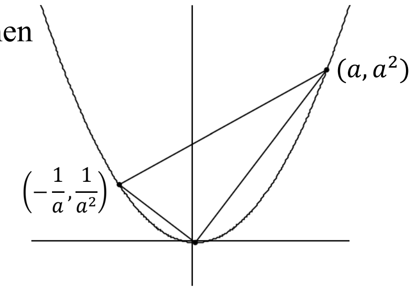
(ii) 200 = 2(G+E+A+R) – 2N.

T	R	I
A	N	G
L	E	2023

Add equations (i) and (ii) to get 400 = 3(G+E+A+R), so G+E+A+R = $\frac{400}{3}$.

(Note that the answer is independent of 2023.)

21. D Let the coordinates of the other vertex of the longer leg be (a, a^2). Then the slope of the longer leg is a, and the slope of the shorter leg is $-\frac{1}{a}$.



The third vertex is the intersection of the line $y = -\frac{1}{a}x$ and $y = x^2$.

Setting the two equations equal and solving for x gives $x = -\frac{1}{a}$.

Thus, the coordinates of the third vertex are $\left(-\frac{1}{a}, \frac{1}{a^2}\right)$.

Using the distance formula, the lengths of the legs are $\sqrt{a^2 + a^4}$ and $\sqrt{\frac{1}{a^2} + \frac{1}{a^4}}$.

Therefore, $\sqrt{a^2 + a^4} \cdot \sqrt{\frac{1}{a^2} + \frac{1}{a^4}} = \frac{10}{3}$. The left side of this equation is

$\sqrt{a^2 + \frac{1}{a^2} + 2} = a + \frac{1}{a} = \frac{a^2+1}{a}$.

Thus, $\frac{a^2+1}{a} = \frac{10}{3}$, from which $a = 3$ or $\frac{1}{3}$. Therefore, the length of the longer leg of the triangle is $\sqrt{90} = 3\sqrt{10}$.

22. E The four-digit palindrome $abba$ can be written as $1001a + 110b$. Then,

$$\frac{1001a+110b}{11a} = 91 + \frac{10b}{a}$$

Thus, $abba$ is divisible by aa if and only if $10b$ is divisible by a. Let's make a chart of the possibilities:

a	1	2	3	4	5	6	7	8	9
# of possibilities for b	10	10	4	5	10	4	2	3	2

Thus, there are 50 such numbers.

23. A **Method 1**: Rewriting the given equation $a^3 + 4a = 8$ as $a^3 = 8 - 4a$ and multiplying by a^4, we get

$$a^7 = 8a^4 - 4a^5 = 8a^4 - 4a^2(a^3) = 8a^4 - 4a^2(8 - 4a) = 8a^4 + 16a^3 - 32a^2.$$

Substituting $a^3 = 8 - 4a$ again into this last equation, we get

$$a^7 = 16a^3 + 8a(a^3) - 32a^2 = 16a^3 + 8a(8 - 4a) - 32a^2 = 16a^3 - 64a^2 + 64a.$$

This last equation can be rewritten asType equation here.

$$a^7 = -64a^2 + 16(8 - 4a) + 64a = -64a^2 + 128.$$

Therefore, $a^7 + 64a^2 = 128$, which is the required value.

Method 2: From $a^3 = 8 - 4a$, Squaring both sides, we have $a^6 = 64 - 64a + 16a^2$ which implies $a^7 = 64a - 64a^2 + 16a^3$. Adding $64a^2$ to both sides, we have;

$a^7 + 64a^2 = 64a + 16a^3 = 16\,(a^3 + 4a) = 16\,(8) = 128$.

24. A **Method 1**: Note first that if $a^* = \frac{a-1}{a+1}$, then $a = \frac{1+a^*}{1-a^*}$

Let $x = n^*$. Then $(n\,^*)^* = \frac{x-1}{x+1} = \tan 15 = \frac{\sin 15}{\cos 15} \Rightarrow (x-1)\cos 15 = (x+1)\sin 15$.

Squaring both sides and rearranging terms, we get

$$x^2(\cos^2 15 - \sin^2 15) - 2x(\cos^2 15 + \sin^2 15) + (\cos^2 15 - \sin^2 15) = 0.$$

Noting that $\cos^2 15 - \sin^2 15 = \cos 30 = \frac{\sqrt{3}}{2}$ and $\cos^2 15 + \sin^2 15 = 1$, this last equation becomes $3x^2 - 4\sqrt{3}x + 3 = 0$. Using the quadratic formula to solve gives $x = \frac{\sqrt{3}}{3}$ and $x = \sqrt{3}$. Using the first value, $\tan 15 = (n\,*)* = \frac{x-1}{x+1} = \frac{\frac{\sqrt{3}}{3}-1}{\frac{\sqrt{3}}{3}+1} = \frac{\sqrt{3}-3}{\sqrt{3}+3}$. But this is negative, while $\tan 15$ is positive. Using $x = \sqrt{3}$, $\tan 15 = (n\,*)* = \frac{x-1}{x+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$, which is positive. Therefore, $n = \frac{1+\sqrt{3}}{1-\sqrt{3}} = -2 - \sqrt{3}$.

Method 2: One can use the double angle formula $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$ with x = 15 degrees to find out that $\tan 15 = 2 - \sqrt{3}$. By repeated substitution, one can work out $(n^*)^* = -\frac{1}{n}$. Then, it remains to solve $-\frac{1}{n} = 2 - \sqrt{3}$ which gives $n = -2 - \sqrt{3}$ by using conjugate.

25. E First observe that the number of five-digit rising numbers is equal to the number of ways to choose 5 different digits from the set {1,2,3,4,5,6,7,8,9}, since there is exactly one rising number for each choice of 5 different digits. So there are ${}_9C_5 = \frac{9!}{5!4!} = 126$ five-digit rising numbers. Of these, ${}_8C_4 = \frac{8!}{4!4!} = 70$ start with 1, ${}_7C_4 = \frac{7!}{4!3!} = 35$ start with 2, ${}_6C_4 = \frac{6!}{4!2!} = 15$ start with 3, ${}_5C_4 = 5$ start with 4, and ${}_4C_4 = 1$ number starts with 5. We need the 100th number in the list, so we need to find the sixth number from the end among those that start with 2. The last (largest) six numbers starting with 2, in decreasing order, are: 26789, 25789, 25689, 25679, 25678, 24789. Thus the 100th number in the list is 24789.