KENNESAW STATE
UNIVERSITY
COLLEGE OF SCIENCE AND MATHEMATICS
Department of Mathematics

THE 2023-2024 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITIONPART II

Calculators are <u>NOT</u> permitted

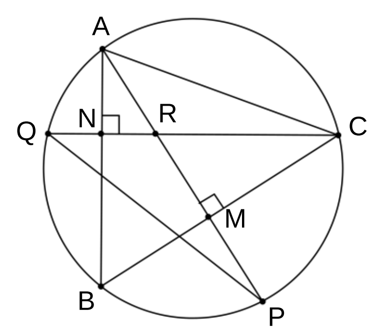
Time allowed: 2 hours

1. Let $P(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where $a_1,\ a_2, \ldots, a_n$ are integers and $a_1 +\ a_2 + \cdots + a_n$ is an even integer. Prove that there are no integer solutions to the equation $P(x) = 0$.

2. Prove that in the set {2420, 2·2420, 3·2420, …, 2024·2420}, there are exactly 44 numbers divisible by 2024.

3. Acute triangle ABC is inscribed in a circle. Altitudes $\overline{\mathrm{AM}}$ and $\overline{\mathrm{CN}}$ meet at point R and are extended to meet the circle at P and Q, respectively. If PQ : AC = 8 : 5, compute, with proof, the value of sin ∠ABC.

   Please include a diagram with your proof.



4. Suppose we have two urns, each containing some red and some blue marbles, with at least one of each color in each urn. Assume that if we choose an urn randomly and then choose a marble randomly from that urn, then the probability of picking a red marble is the same as we would get by combining all the marbles into one urn and choosing a marble from that one at random. If the first urn contains 7 marbles and the second one contains 5 red marbles, find all possibilities for the number of marbles in the second urn. Prove that your answer is correct.

5. Prove that the hands of a clock (hour, minute, second hand) never trisect the face of the clock.

Solutions

1. Method 1:
If $P(b) = 0$ for some integer b, then
$1 + a_1 b + a_2 b^2 + \cdots + a_n b^n = 0 \;\Rightarrow\; b(a_1 + a_2 b + \cdots + a_n b^{n-1}) = -1.$

But this implies that b divides -1 and the only integers that can do that are -1 and 1.

If $b = 1$, then $P(b) = 1 + a_1 + a_2 + \ldots + a_n$ which is odd and so cannot be 0.
If $b = -1$, then $P(b) = 1 - a_1 + a_2 - \ldots + (-1)^n a_n$
$= (1 + a_1 + a_2 + \ldots + a_n) - 2(a_1 + a_3 + a_5 + \cdots)$
which is also odd and so cannot be 0. Therefore, there are no integer solutions to $P(x) = 0$.

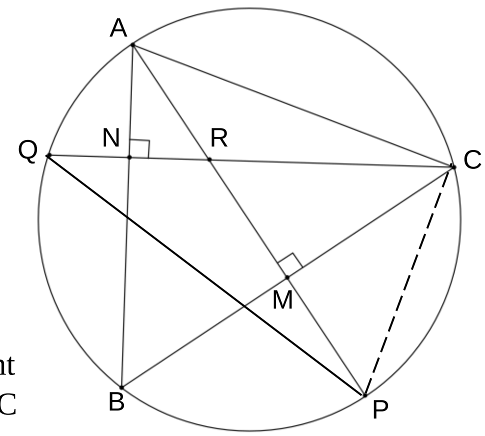
Method 2:
If $P(b) = 0$ for some integer b:
If b is even, then $1 + a_1 b + a_2 b^2 + \cdots + a_n b^n$, cannot be zero.
If b is odd, $b = 2k + 1$ then $P(2k + 1) = 1 + 2km + a_1 + a_2 + \cdots + a_n$, cannot be zero.

2. Since $(2024, 2420) = 44$, then
$2024 = 44s$ and $2420 = 44r$ for some integers r and s where $(r, s) = 1$ (i.e. r and s are relatively prime).
Noting that $\frac{2420}{2024} = \frac{r}{s}$, if we divide each of the numbers in the given set by *2024*. We obtain the quotients:

$$\left\{\frac{2420}{2024}, \frac{2(2420)}{2024}, \ldots, \frac{2024(2420)}{2024}\right\} = \frac{r}{s}, \frac{2r}{s}, \ldots, \frac{(2024)r}{s} = \frac{r}{s}, \frac{2r}{s}, \ldots, \frac{(s)r}{s}, \ldots, \frac{(2s)r}{s}, \ldots, \frac{(44s)r}{s}.$$

Since r and s are relatively prime, the only integers among the numbers in the above set are the quotients in which the coefficient of r in the numerator is a multiple of s.
Since $2024 = 44$s, this happens *44* times as the coefficients go through the values 1, 2, …, $44s=2024$.

3. Let R be the intersection of altitudes $\overline{\mathrm{AM}}$ and $\overline{\mathrm{CN}}$.
ΔARC ~ ΔQRP since angles ARC and QRP are congruent vertical angles and inscribed angles PAC and PQC intercept the same arc PC. Therefore, $\frac{\mathrm{PR}}{\mathrm{RC}} = \frac{\mathrm{PQ}}{\mathrm{AC}} = \frac{8}{5}$. Also, right triangles ANR and CMR are similar because $\angle$ARN ≅ $\angle$CRM. Therefore, $\angle$MCR ≅ $\angle$NAR and since these are inscribed angles, arc BQ is congruent to arc BP.



Now, construct $\overline{\mathrm{PC}}$. Inscribed angles QCB and PCB are congruent because they intercept congruent arcs. Therefore, ΔRMC ≅ ΔPMC and $\overline{\mathrm{RM}} \cong \overline{\mathrm{PM}}$ and $\overline{\mathrm{RC}} \cong \overline{\mathrm{PC}}$ Thus, $\frac{\mathrm{MP}}{\mathrm{PC}} = \frac{\frac{1}{2}(\mathrm{PR})}{\mathrm{RC}} = \frac{4}{5}$, making ΔMPC a 3-4-5 right triangle, and sin $\angle$CPM = $\frac{3}{5}$. Finally, since inscribed angles CPM and ABC intercept the same arc, they are congruent and sin $\angle$ABC = $\frac{3}{5}$.

4. Denote the number of red marbles in the first urn by r and the total number of marbles in the second by t. If we choose a marble randomly from the first urn then the probability of getting a red one is $\frac{r}{7}$, while this probability is $\frac{5}{t}$ for the second urn. Now we begin by choosing our urn at random, so we have $\frac{1}{2}$ probability of choosing from the first and $\frac{1}{2}$ of choosing from the second urn. This means that the probability of ending up with a red marble is $\frac{1}{2}\left(\frac{r}{7}+\frac{5}{t}\right)$.
On the other hand, if we combine all the marbles into one urn, then we have $r + 5$ red marbles out of a total $7 + t$, so the probability of getting a red marble in this case is $\frac{r+5}{7+t}$. If these two probabilities are equal, then:

$$0 = \frac{1}{2}\left(\frac{r}{7}+\frac{5}{t}\right) - \frac{r+5}{7+t} = \frac{245-35t-7rt+rt^2}{14t(7+t)} = \frac{(t-7)(rt-35)}{14t(7+t)}.$$

The expression on the right can only be zero if $t = 7$ or $rt = 35$. Now $1 \le r \le 6$, since both colors are represented in the first urn, so the second equation yields $r = 1$, $t = 35$ or $r = 5$, $t = 7$. Therefore, the only possibilities for t are 7 and 35. Thus, we can have 7 marbles in the second urn (in which case the number of red marbles in the first urn can be any value between 1 and 6), or we can have 35 marbles in the second urn (in which case there is exactly 1 red in the first urn).

5. <u>Method 1:</u> Let h, m, and s denote the hour hand, minute hand, and second hand, respectively. In order to avoid confusion, we will refer to the distance the minute hand travels each minute as a unit. If the hands trisect the face, then there are only two possible cases. Either m is 20 units ahead of h (Figure A), or m is 20 units behind h (Figure B)

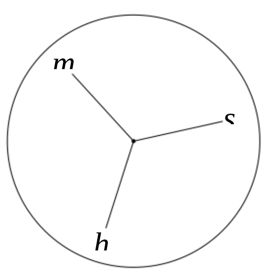


Figure A

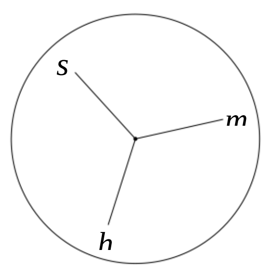


Figure B

Begin at exactly 12:00 AM (when all three hands overlap). When the hands trisect the face, if h has moved x units, m has moved $12x$ units, and s moved $60(12x) = 720x$ units.

For Figure A, the amount moved by s plus 20 units will be k turns more than the amount moved by h. Thus, $720x + 20 = x + 60k \Rightarrow 719x + 20 = 60k$.
The amount moved by m subtract 20 units will be n turns more than the amount moved by h. Thus,
(2) $12x - 20 = x + 60n \Rightarrow 11x - 20 = 60n$.
Multiplying equation (1) by 11 and equation (2) by 719 and subtracting, we eliminate x and we eventually obtain $730 = 3(11k - 719n)$. But this says that 3 is a factor of 730, which is false. Therefore, the hands cannot trisect the face in Figure A.
For Figure B, all we have to do is change the signs of 20 in both equations and we reach the same conclusion.

Method 2:
The rate at which the angle between two hands of the clock changes is constant. As a result: if we see that at a time T hours later than 12:00, the minute hand is 120 degrees past the hour hand, then at a time 2T hours later than 12:00, the minute hand is 240 degrees past the hour hand, and at a time 3T hours later than 12:00, the minute hand is 360 degrees past the hour hand: they meet again.

Similarly, if at a time T hours later than 12:00, the second hand is 120 degrees past the minute hand, then at a time 3T hours later than 12:00, the second hand is 360 degrees past the minute hand - and they also meet. So, if we see Figure A (or, by a similar argument, Figure B) at time T, then at time 3T, we must see all three hands in the same place. This only happens every 12 hours.

Notice that the minute hand moves 12 times faster than the hour hand, so it passes the hour hand 11 times over the course of 12 hours: every time the hour hand makes 1/11 of a circle. Over that time, the second hand makes 65 full turns and another 5/11 of a circle, so it advances by 4/11 of a circle relative to the other two hands every time they meet. This has to repeat 11 times for the second hand to meet the other two, at which point it's 12:00 again.

Since 3T is a multiple of 12 hours, T must be a multiple of 4 hours, and we only need to investigate two times to see if they look like Figure A or Figure B: we need to check 4:00 and 8:00. However, we don't see Figure A or Figure B at those times: instead, the second hand and the minute hand meet at the top of the clock. Therefore, we never see Figure A or Figure B.